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A TREATISE

ON THE

ELEMENTS OF ALGEBRA.

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A TREATISE

ON THE

ELEMENTS OF ALGEBRA:

FOR THE USE OF

Schools and Private Students.

BY

G. AINSWORTH, B.A., AND J. YEATS, F.R.G.S.

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1854.

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PREFACE.

In presenting to the public the following Treatise on Algebra as one of the Educational Series, we premise that great pains have been taken to render it efficient and comprehensive, without being diffuse.

The text has been revised by eminent Teachers, and the exercises, numerous and practical, have been selected chiefly from the best modern school-books of the Continent. The admirable works of Grunert, Heis, Lagerwij, and Strootman have been frequently consulted in the compilation, and to them our acknowledgments and thanks are respectfully tendered.

Peckham, Dec. 1854.

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THE

ELEMENTS OF ALGEBRA.

CHAPTER I.

SIGNS, SYMBOLS, ETC.

ALGEBRA affords the double advantage of a substitution of letters for quantities at will, and the use of figures as auxiliaries to those letters. Thus if a represent the number of solid inches in a solid foot, 1728, then $8a$ will represent 8 times 1728, or 13,824. The algebraic expression would be, since $a=1728$, therefore $8a=13{,}824$.

Known values or magnitudes are usually denoted by the first letters of the alphabet, a, b, c; and unknown ones by x, y, z. Occasionally, too, the Greek characters α, β, γ, δ,—θ, ϕ, χ, are similarly employed.

The relation of quantities to each other, or their connection, is expressed by symbols. These are, $=$, $+$, $-$, $\times$, $\div$; and besides them a few abbreviations are in use. An explanation of both is necessary.

Whenever one quantity is to be noted as exactly equal to another, the sign of equality ($=$) must be put between them. Thus if $a=4$, and $b=3$, $3a=4b$; or three times $a=4$ times b.

Whenever one quantity is to be added to another, the sign $+$ (plus) must precede the quantity added. Thus $a+b$ signifies that b is to be added to a; then supposing the previous values to be retained, $a+b=7$.

Whenever a quantity is to be taken from another, the sign $-$ (minus) must precede the subtrahend. Thus $a-b$ signifies that b is to be taken from a: hence, supposing the values as before, $a-b=1$.

Whenever a smaller quantity is to be taken from a larger one, and it is not known which of the two is the greater, the sign $\sim$ is used to denote the difference between them.

$>$ signifies *greater than*; $<$ *less than*.

Any quantity immediately preceded by the sign $+$ is said to be *positive*.

Any quantity immediately preceded by the sign $-$ is said to be *negative*.

A quantity preceded by neither $+$ nor $-$ is always considered *positive*. Thus a signifies $+a$.

When the product of two quantities is sought, the sign of multiplication (× or .) must be placed between them. Ex.: $a \times b$ signifies a multiplied by b; and $a \, . \, b$ the same. But the most common mode of expressing the product of two letters is by simply joining them together: thus, ab. Hence, in the preceding example, we should write $a \times b = ab$; and supposing the values to remain as before, the product would be $4 \times 3 = 12$.

The figure placed before any algebraical letter is called its *coefficient*. Thus in the expressions $3a$, $4b$, 3 is the coefficient of a, and 4 is the coefficient of b. Coefficients may be positive or negative. Of course every letter will have 1 as coefficient, if no higher number; but the 1 is usually omitted in writing. Thus ac means once the quantity a multiplied into once the quantity c.

Division is usually denoted by placing the dividend above the divisor, and drawing a line between them in fractional form. Thus $\frac{a}{b}$ denotes that the quantity a is to be divided by the quantity b. Frequently, too, the sign ÷ is made use of as a mark of division, in which case it precedes the divisor and follows the dividend. $a \div c$ and $\frac{a}{c}$ denote the same thing. If $a=4$, $b=3$, and $c=2$, then $\frac{2a}{c} = \frac{8}{2} = 4$; $\frac{8a-4b}{2c} = \frac{32-12}{4} = 5$.

$\because$ signifies *since* or *because*.

$\therefore$ signifies *therefore* or *consequently*; hence, supposing $a=6$, $b=4$, $c=3$, and $d=8$, we may say, $\because ab=24$ and $cd=24$, $\therefore ab=cd$.

A small figure placed to the right hand of a letter or quantity, and above the line, is called the index or exponent to the power of that quantity. Thus in a^4, 4 is the index to a. The expression is read a to the 4th power, and signifies a multiplied into itself 4 times. Supposing the value of a to be 3, $a^4 = aaaa$, or $3 \times 3 \times 3 \times 3 = 81$.

The different powers of a quantity are thus read:

a (or a^1, which is never written) a to the first power, or simply a.
a^2, the square of a, or a to the second power.
a^3, the cube of a, or a to the third power.
a^4, the fourth power of a, or a to the fourth power, &c., &c.

When a quantity is preceded by the sign $\sqrt{\ }$, some root of that quantity is denoted. When a small figure is placed to the left hand of the sign and above the line, it is called the index or exponent to the root, and it shows what root is to be extracted: thus $\sqrt[3]{c}$ denotes the cube root of c.

In expressing the *square root*, the index 2 is omitted. The index of a root shows how often that root has to be multiplied into itself to equal the quantity from which it was extracted: thus, supposing $a=100$, then $\sqrt{a}=10$, $\because 10 \times 10 = 100$; supposing $b=1728$, then $\sqrt[3]{b}=12$, $\because 12 \times 12 \times 12 = 1728$; supposing $c=81$, then $\sqrt[4]{c}=3$, $\because 3 \times 3 \times 3 \times 3 = 81$.

The roots of quantities are thus read:

$\sqrt{a}$ is the square root of a.
$\sqrt[3]{b}$ is the cube root of b.
$\sqrt[4]{c}$ is the fourth root of c, &c. &c.

Similar quantities, or like quantities, are those which contain the same combinations of letters and powers of letters. Dissimilar or unlike quantities are those which do not contain the same combinations of letters, or letters of like powers. Thus $3a+5a-2a$ are similar quantities, because they all show the same letter, and the same power of it. $5a^3b^2c-3a^3b^2c$ are also similar quantities, and for the same reason; but $4a^2b^3-3a^3b^2+c$ are dissimilar quantities, because, though the letters are the same, the powers are different.

When several letters or combinations of letters go to form but one quantity, they are united by a vinculum, or placed within brackets; $x-\overline{y-z}$ is the same as $x-(y-z)$.

A single instance will suffice to show that great care is required in the use of these brackets. In the example last quoted, $x-(y-z)$ is not the same quantity as $x-y-z$. $x-(y-z)$ signifies that x is to be diminished, not by y, but by $y-z$, or the difference between y and z; whereas $x-y-z$ means that x is to be diminished by y and by z also, or by the sum of these two.

Hence, if $x=4$, $y=3$, $z=1$, we have

$x-y-z=4-3-1=0.$ $\quad$ $x-(y-z)=4-2=2.$

$2x-3y+2z=8-9+2=1.$ $\quad$ $2x-(3y+2z)=8-11=-3.$

$2x+y-z=8+3-1=10.$ $\quad$ $2(x+y)-z=14-1=13.$

The vinculum is employed mostly in fractional expressions, and in connection with the radical sign ($\surd$) before compound quantities. The bracket unites quantities chiefly for multiplication: $\frac{x+y+z}{4}$ implies that the whole quantity $x+y+z$ is to be divided by 4, and $4(x+y+z)$ that the same quantity is to be multiplied by 4.

DEFINITIONS.

The quantities employed in an operation of addition or of subtraction are called *terms*. Thus in $2a-4a+5a=3a$, the terms are $2a$, $-4a$, $+5a$.

The quantities involved in an operation of multiplication are called *factors*. Thus in the expression $a\times b\times c=abc$, the factors are a, b, and c.

An algebraical expression is called *simple* if it consists of but one term, and *compound* if of more than one. Thus $2a$, $-4a$, and 5, are each simple quantities; but the whole taken together $(2a-4a+5)$ forms a compound quantity.

A quantity expressed by a single term, as a, is called a monomial; one expressed by two terms, as $a+b$, a binomial; one by three terms, as $a+b-c$, a trinomial; and one by more than these a multinomial.

Quantities, whether terms or factors, expressed by different letters have those letters arranged as nearly as possible in the order of the alphabet. This is done for the sake of method merely. $a+b+c$ is the same as $b+a+c$, or $c+a+b$; and $x\times y\times z=z\times x\times y=xyz$.

As the distinction between terms and factors is important, a few examples and further illustrations are given :

I. $x+y-z$ is a trinomial, composed of three terms, x, y, and z.

II. $x+(y-z)$ is a binomial, composed of but two terms, x and $(y-z)$.

III. $2x-(y+z)$ is a binomial, composed of two terms, $2x$ and $-(y+z)$.

IV. $x+3(y+z)$ is a binomial, composed of two terms, x and $3(y+z)$.

In Example III. $2x$ is a term composed of two factors, 2 and x.

In Example IV. $3(y+z)$ is a term composed of two factors, 3 and $(y+z)$; the factor $(y+z)$ being itself a compound quantity of two terms.

V. $3(x-y+z)$ is a single term, or a simple quantity composed of two factors, 3, and $(x-y+z)$, of which the latter is a quantity compounded of three terms.

VI. x^2 is a single term composed of two factors, x and x; ax^2 is also a single term composed of three factors, a and x and x; $4ax^2y$ might also be a single term composed of the five factors 4, a, x, x, and y, though it is not usual to resolve *powers* such as x^2 into their elementary factors unless absolutely requisite.

EXERCISE I.

If $a=5$, $b=3$, $c=4$, $d=2$, $e=1$, and $f=0$, find the numerical values of the following expressions :

1. $2a-4c+2e+5b-d$.
2. $3b+7e-4f+6a-4b-c$.
3. $c+b+3e-4a+6f-2c+d$.
4. $3f-c+2a+3d-4c+e$.
5. $6b+a+4d-f+3e-6c$.
6. $7a+3f-8c+3b-4d+3e$.

If $a=7$, $b=5$, $c=0$, $d=6$, $e=4$, $f=8$, $g=2$, what are the numerical values of the following?

7. $2b-a+d+3c-2e-f+4g$.
8. $3d-4c+g-7a+8e+3b-3f$.
9. $2c+d+e+3f-3a+4b-g$.
10. $6a-7b+2g-4c+2d-f+2e$.
11. $3e+4g-f+4a-6c+2b-3d$.
12. $8b-4a+6c+7d-e+4f-3g$.

EXERCISE II.

If $a=4$, $b=2$, $c=5$, $d=3$, and $e=0$, what are the values of

1. $\frac{ab}{d-b}$.
2. $\frac{2bc-a}{ab}$.
3. $\frac{2a}{b}+\frac{3c}{d}$.
4. $\frac{4b}{a}-\frac{4c}{2b}+\frac{2a}{b}$.
5. $\frac{4b}{a}+\frac{3c}{d}-\frac{a}{b}$.
6. $\frac{5d}{c}-\frac{2c}{b}+\frac{6a}{d}$.
7. $\frac{2b-d}{e}+\frac{4c-5b}{2c}+c$.
8. $\frac{ac+bd-3c}{d-b}$.
9. $\frac{b^2+2a^3-d^3+a^2}{c^2}+cd$.
10. $2ab^2-c^2+3ad$.
11. $6b^2d+ac^2-\frac{2a^2c}{ac+5b^2}$.
12. $\frac{abc+de}{2b-a+c}+\frac{b^2c^3d}{ac+5b}-\frac{3ab+4c-da}{c+d}$.

EXERCISE III.

What are the numerical values of

1. $\sqrt{9}+2\sqrt{36}-\sqrt{16}$.
2. $2\sqrt{25}+4\sqrt{4}-5\sqrt{4}$.
3. $\sqrt{81}-2\sqrt[5]{32}+2\sqrt[3]{64}$.
4. $\sqrt[3]{8}+3\sqrt{49}+3\sqrt{16}$.
5. $\sqrt{16}-4\sqrt[3]{1}+2\sqrt{9}+2\sqrt[3]{125}$.
6. $\sqrt[4]{1}+3\sqrt{16}-\sqrt[3]{8}+3\sqrt{49}-2\sqrt{81}$.

If $a=16$, $b=36$, $c=9$, $d=4$, $e=1$, what are the values of

7. $4\sqrt{d}+2\sqrt{b}-\sqrt{16e}$.
8. $\sqrt{4b}+3\sqrt{c}-3\sqrt[4]{ae}$.
9. $\sqrt{a^2}+5\sqrt{ad}-\sqrt{c}$.
10. $\sqrt{bd}-2\sqrt{a}+\frac{3\sqrt{b}}{e}+\sqrt{ce}$.
11. $4\sqrt{9c}+3\sqrt{d}-\sqrt{25d}+\frac{6\sqrt{a}}{\sqrt{be}}$.
12. $\sqrt{ae}-\sqrt{b}+9\sqrt{cd}-2\sqrt{ca}$.

EXERCISE IV.

1. Show the difference between $(a-b)+c$; $a-(b+c)$; $a-(b-c)$; $a-b-c$; and give the value of each when $a=36$, $b=17$, $c=2$; or when $a=6336$, $b=1719$, and $c=287$.

2. Work out $12-7-(2+1)$; $12-7-2+1$; $12-(7-2+1)$; $12-(7-2)+1$; $12-\{7-(2+1)\}$.

3. If $m=565$, $n=217$, $o=181$, $p=45$, $q=57$, what will be the value of $m-n-o-p+q$; $m-(n-o)-(p+q)$; $m-\{n-(o-p)+q\}$; $m-\{n-o-(p+q)\}$; $m-\{n-(o-p+q)\}$; $m-n-(o-p+q)$; and $m-\{n-\{o-(p+q)\}\}$?

4. In what manner must the brackets be applied so that from $m-n+p-q$ shall arise the values 22, 44, and 66, when $m=88$, $n=33$, $p=11$, $q=22$?

5. In the expression $a-(b-c)$ substitute $m+n$ for a; $p-(q-r)$ for b; and $x+y$ for c.

6. When $a=200$, $b=21$, $c=7$, $d=6$, what is the value of the following; $a(b-c+d)$; $ab-c+d$; $a\{b-(c+d)\}$; $(a-b)(c-d)$; $(a-b)c-d$; $a-b(c-d)$; $a-bc-d$; $(a+b-c)d$; $a+b-cd$; $a+(b-c)d$?

7. In the expression $b+bc-c$, substitute $m+n$ for b, and $p-q$ for c.

8. If $a=4$, $b=5$, $c=6$, $d=7$, what is the value of $(ab)\times c$; $a\times(bc)$; $a\times(bc)\times d$; $a(bcd)$; $(ab)\times(cd)$; $a\{b(cd)\}$; $abcd$; $a^2(a^3b)b^4$?

9. If $a=1050$, $b=420$, $c=105$, $d=21$, what is the value of $(a-b+c)d$; $(a-b)(c+d)$; $a-(b+c)d$; $\{a-(b+c)d\}$; $\frac{a-b}{c+d}$; $\frac{a-(b+c)}{d}$?

EXERCISE V.

Miscellaneous Examples.

Write down

1. The sum of a and b.
2. The difference of a and b.
3. The sum of a, b, and c.
4. The product of a and b.
5. The product of a, b, and c.
6. The difference of a and b, divided by their sum.
7. The sum of a, b, and c, divided by a.
8. The product of x, y, and z, divided by their sum.
9. The product of a, b, c, and d, divided by the sum of a and b.
10. Is the sum of a, b, c, and d, divided by the sum of c and d, properly expressed in the following manner $\frac{d+c+b+a}{dc}$?
11. Why is the product of a, b, x, and y incorrectly expressed thus, $yxba$?
12. Write the sum of a and b, multiplied by c.
13. The difference of a and b, multiplied by c.
14. The product of a and b + the product of c and d.
15. The product of a and b — the product of c and d.

16. Is the product of a, b, c, and d, divided by their sum, correctly expressed thus, $\frac{a+b+c+d}{dcba}$?

17. Express the product of x and y + the product of p and q, and the sum of these quantities divided by the sum of x and y.

18. Express 6 times the sum of a and b, divided by the sum of c and d.

19. 8 times the difference of a and b, divided by the product of c and d.

20. a and b multiplied by their product.

21. The product of a, b, and c, divided by the sum of a and b, less c.

22. If $a=2$, $b=3$, and $c=4$, what is the value of $(a+b)c$?

23. If $a=8$, $b=12$, $c=5$, and $d=3$, what is the value of $\frac{a+b-c}{d}$?

24. If $a=5$, $b=6$, and $c=7$, what is the value of $3(a+2b+3c)$?

25. If $a=8$, $b=12$, $c=16$, and $d=24$, what are the values of $\frac{a+b+c+d}{ab}$, $\frac{a+b+c+d}{bc}$, $\frac{4(a+b+c+d)}{cd}$, and $\frac{\frac{1}{8}(a+b+c+d)}{ad}$?

26. Said one boy to another, "If $a=4$, $b=9$, and $c=5$, then our joint stock of marbles amounts to $\frac{abc}{a+b+c}$." How many were there?

27. His comrade replied, "We ought to have $\frac{a^2bc}{a+b-c}$." What is that number?

28. The assignees of an estate reported that the debts were $\frac{abcd}{a+b+c+d}$ and that the eighth part might be paid off; supposing $a=3$, $b=4$, $c=5$, and $d=6$, what sum remained unpaid?

29. A bill amounted to $a^2+2ab+b^2$ pounds. How much was that, if $a=5$ and $b=7$?

30. Could it be paid by $a^2-2ab+b^2$ pounds, if $a=16$ and $b=4$?

31. An officer's monthly pay was $\frac{1}{4}(a^2+2ab+b^2)$ pounds, and his annual expenditure $54\left(\frac{a+b+c+d}{a^2}\right)$; did he save any thing, if $a=3$, $b=4$, $c=5$, and $d=6$?

32. A tavern bill was jocularly said to amount to $\frac{ab+bc+ac}{2a+b+c}$ shillings, and three had to pay it; what was each person's share, supposing $a=6$, $b=8$, and $c=12$?

33. Assuming the same values to the letters, one of the three found that his shoemaker's bill was $\frac{36(a+2b+3c)}{ac}$, and his tailor's $\frac{ab+bc+ac}{a(b+c)}$; to what did they amount together?

34. A gentleman said, that if $a=£60$, $b=£1200$, $c=£1000$, $d=£40$, and $e=£560$, then his property was worth $\frac{e(a+b+c+d)}{a+d}$; what was the value of it?

35. His debts, on the preceding supposition, were only $\frac{(a+b+c+d)e}{ad}$ pounds; when they were paid off, what did he really possess?

36. If $a=80$, $b=60$, $c=40$, and $d=20$, how much more is $\frac{ab+bc+cd+ad}{(a-b)\,(c-d)}$ than $\frac{(a+b+c)d}{(a+b)+(c-d)}$?

37. A boy expressed his age by saying, that if $a=6$, $b=5$, and $c=7$, he was $\frac{6(a^2+b^2+c^2)}{a+ac}$ years old; how old was he?

38. His grandfather said, "Well, at that rate, I must be $\frac{33abc}{a^2+b^2+c^2}$." How much older was he than his grandson?

39. A house was bought for $\frac{16(a+b)+5(c+d)}{2}$ pounds; if $a=60$, $b=70$, $c=90$, and $d=120$, what was the cost?

40. The expenses of conveyancing were $\frac{\frac{1}{5}(a+b)+\frac{2}{5}(c+d)}{\frac{1}{5}}$; to what did they amount, at the same rate?

CHAPTER II.

ADDITION.

In the addition of like quantities, to the difference of the sums of the positive and negative coefficients prefix the sign of the greater, and to it attach the proper letter or letters.

Thus,

$$\begin{array}{r}3a-12b-7c^2+5d+8e-f\\ -5a-4b-5c^2+7d-4e+m\\ 8a+13b-3c^2-d+7e-z^2\\ \hline 6a-3b-15c^2+11(d+e)-f+m-z^2.\end{array}$$

The addition of unlike quantities is performed by merely arranging the different expressions in alphabetical order, and connecting them by their proper signs.

The student is advised to make use of brackets as often as possible in expressing the sums of quantities.

Exercise VI.

1. $2a+a$.
2. $9a+15a$.
3. $7b^2+3b^2$.
4. $5a^2b+9a^2b-8a^2b-3a^2b$.
5. $10d-7d$.
6. $x+13x-8x+7x-2x$.
7. $5a-b+2c-4d+6e$.
8. $5m-3m-2n$.

9. $6(a+x)+5(a+x)-7(a+x)+(a+x)-2(a+x)$.

10. $(a-y)+12(a-y)-5(a-y)+2(a-y)-9(a-y)$.

11. $ax+bx$.

12. $5a+4b-3c-7d+8+3a-12b+7c-10d-4$.

13. $12h-3c-7f+3g-3h+8c-2f-9g+5x$.

14. $16a-5b+10c-9d+3a+18b-5c-7d+3e-7a-2b-3d+5e-9h+11a-3b+2c+8d+7h$.

15. $12a^2-6bc+8c^2-f+13-7a^2-3bc+5c^2+9f-9+5a^2+18bc-11c^2-3f+10+4bc-6c^2+f-11$.

16. $-4g+8h-k+3l+7+7g-5h+11k-l-3+g+h-12k+8l-10$.

17. $15x-12y+3z-v-3x+4y-z+8v+x+7z-v-3y-4z+9v-2z+5v$.

18. $7p+3q-v+s+3p+5q-3v+7s-6p+q-s-6q+5v+3s+v+5s$.

19. $6a-4b+3c+6b+4d-4c+3a+a-d+2c+2b+c-b+e+5a+2d$.

20. $3a^2b-7ab^2+xy+7xy-2a^2b-ab^2+2a^2b+7y^2+2b^3+2a^3-xy+3ab^2-a^3-7b^3-xy-x^2-7xy+2y^2$.

21. $3ax+4by+2z+7cx-3dy-5z+8cx-y-z$.

22. $3\sqrt{x}+4\sqrt{y}+3\sqrt{z}+5\sqrt{y}-2\sqrt{x}-8\sqrt{z}+10\sqrt{z}-8\sqrt{x}+\sqrt{y}+\sqrt{x}-7\sqrt{y}+15\sqrt{z}$.

23. $12ax^3-17a^2x^2+9a^2x^2-1\tfrac{1}{2}a^3x+7a^3x-7\tfrac{1}{2}ax^3-a^4+3a^4+1\tfrac{3}{4}a^3x+8a^2x^3+9a^3x-4\tfrac{1}{2}ax^3-2a^4$.

24. $-xyz+x^3-xz^3+x^2z+2x^2y+5y^3-7xy^2+3xz^2-9y^3-x^2z+8xy^2+7x^2y+3y^3+5xz^2+3xyz-7x^2z-8y^3-3xy^2-xz^2+z^3-9x^2z-2xyz+xz^2-3x^2z+y^3$.

SUBTRACTION.

Rule. Change signs in the subtrahend, and then proceed as in addition. Suppose that b has to be taken from a: following the rule, the expression will be written $a-b$, and the reason is clear.

But supposing that $b-c$ has to be taken from a, the subtrahend must become, according to the rule, $-b+c$, the whole expression being $a-b+c$. And the reason of this, if not apparent at once, is evident after a little consideration. For as a is not to be diminished by b, but by $b-c$ (that is by c less than b), so $a-b$ and $-c$ *also*, would be a diminution too great by c. Let $a=8$, $b=6$, and $c=4$, then $a-b=8-6$, $=2$; $a-b-c=8-6-4$, $=8-10$, $=-2$; but $a-(b-c)$ or $a-b+c$ (with the sign changed) $=8-6+4$, or $12-6=6$.

The reason for changing signs may also be illustrated thus: $x=x-y+y$; and if we subtract $+y$ from x, $x-y$ remains, just as if $-y$ had been added to x according to the rule. Again, $x=x+y-y$, and if $-y$ be subtracted from x, $x+y$ remains, as though $+y$ had been added to it, according to the rule.

Very often there will seem to be no necessity for changing signs in subtraction, since by simply deducting the smaller of two coefficients from the larger, and annexing the proper letter, the end will be answered; thus in taking $3a$ from $5a$, the remainder is seen at a glance to be $2a$; but in reality the rule still comes into operation, for $3a$ and $5a$ are both + in the first instance, and the sign of $+3a$ must be changed into $-3a$ before the result $+2a$ is obtained.

EXERCISE VII.

1. Take $-8a$ from $-16a$; $-12ac$ from $-18ac$; $-3ab$ from $-11ab$; $-5xyz$ from $-16xyz$.

2. Take $-5(a+d)$ from $-16(a+d)$; $-14(x+y+z)$ from $-21(x+y+z)$; $-15a(bc+dc)$ from $-32a(bc+dc)$; $-a(b+c)$ from $-3a(b+c)$.

3. How much is a more than $-a$? $2a$ more than $-2a$? $3b$ more than $-3b$? $4x$ more than $-4x$?

4. How much is $5ac$ more than $-5ac$? $6abc$ more than $-6abc$? $7(x-y)$ more than $-7(x-y)$? $8(x+y+z)$ more than $-8(x+y+z)$?

5. How much is $16a$ more than $-2a$? $32ab$ more than $-15ab$? ab more than $-11ab$? $3abcd$ more than $-12abcd$? $5(a+e)$ more than $-12(a+e)$? $16(x+z)$ more than $-(x+z)$?

6. Take $3a+4b$ from $11a+5b$; $16x+y$ from $25x+16y$.

7. Take $24xy+16z$ from $25xy+20z$; $4ab+2ac$ from $16ab+3ac$.

8. Take $4a$ from 0; $6b$ from 0; $16ac$ from 0; $32x$ from 0; $3abc$ from 0; $4(x+y)$ from 0; $5(a+b)$ from 0.

9. How much is $4a-(3a+b)$? $16-(11b+5c)$? $3x-(x+4y)$? $2xy-(xy+yz)$? $4xyz-(3xyz+5uv)$?

10. Take -4 from 0; $-a$ from 0; $-b$ from 0; $-3a$ from 0; $-6ab$ from 0; $7xyz$ from 0; $3(x+y)$ from 0.

11. How much is $3a-(a-3b)$? $6ab-(4ab-3c)$? $16xy-(11xy-5z)$? $3xyz-(xyz-4a)$? $5abc-(2abc-4d)$? $12a^2-(6a^2-4b^2)$?

12. How much is the difference between $a+b$ and $a-b$?

13. How much is the difference between $3a+4b$ and $3a-4b$?

14. Take $a-b+c$ from $2a+b-c$; $4b-3c$ from $3b+4c$; $x+y-z$ from $-x+y+z$.

15. Take $3a$ from $5b$; $5x$ from $6y$; $3z$ from $7x$; $5ab$ from $3cd$; $3ac$ from $-5xy$; $-5xy$ from $3ab$; $-5x$ from $12xyz$.

16. How much is $3a-(-4a)$? $5b-(-3b)$? $16ac-(-5ac)$? $11xyz-(-3xyz)$?

17. How much is $a(x+y)$ more than $b(x+y)$? $4(a+b)$ more than $x(a+b)$? $3a(b+c)$ more than $4b(b+c)$? $5p(q+r)$ more than $-3q(q+r)$? $6a(c+d)$ more than $-5d(c+d)$?

18. Subtract $5a-7b+3f$ from $11a-5b+5f$; and $11a-5b+5f$ from $5a-7b+3f$.

19. Subtract $a+2b-3c$ from $3a-2b+c$; and $3a-2b+c$ from $a+2b-3c$.

20. Subtract $5a+3c+5e-11g$ from $16a+5b-7d+3f$; and $-16a+5b-7d-3f$ from $5a+3c+5e-11g$.

21. How much is $3x+5y-6z$ more than $-3x-4y+6z$; and $-3x-4y+6z$ more than $3x+5y-6z$?

22. From the sum of $3a-4b$, $6a+2b$, $-3a+7b$, and $5a-3b$ subtract the sum of $4a+6b$, $3a-2b$, and $-12a+5b$.

23. Required the sum of the difference of $(7c+8d)-(12c-7d)+$ the difference of $(12c+5d)-(8c-4d)$.

24. How much is $(6c-4ef)-(12c+ef)$ greater than $(8e+8ef)-(-3c+ef)$?

25. Subtract the sum of $(a+b)+(a-b)$ from $(a+b)-(-a+b)$.

26. How much is $(7x+12y)-(-8x+4y)$ greater than $(3x+y)+(4x-y)+(-12x+5y)$?

27. How much is $a-b$ less than $a-(-b)$?

28. If from $a(x+y)$, $b(x+y)$ is subtracted, and then again $c(x+y)$ added, what is the sum?

29. From $3a^3-7a^2b+4ab^2-3b^3$ take $9a^3-a^2b+2ab^2-2b^3$.

30. From $7x\sqrt{y}+9y\sqrt{y}-11\sqrt{xy}$ subtract $-2x\sqrt{xy}+2x\sqrt{y}-y\sqrt{y}+12a$.

31. From $13a\sqrt{(x-y)}-17b\sqrt{(x+y)}-axy$ subtract $17b\sqrt{(x-y)}+bxy-x^2+a^2y$.

32. From the sum of $8a^3-8a^2+17$, $8a^2-18+7a$, and $6a-a^2-7a^3-11$ subtract the sum of $10a^2-3a-16$, $a-a^2+a^3$, and $8a^2+2a+18$.

It has been said that when several letters or combinations of letters go to form but one quantity, they are united by a vinculum, or placed within brackets. Thus $x-\overline{y-z}$ is the same as $x-(y-z)$.

Further instructions in the use of brackets will now be given.

In the addition, subtraction, multiplication, and division of simple quantities, brackets are not needed, so long as the quantities are taken up in alphabetical order: *e.g.* instead of $\{(a+b)+c\}+d$, we write $a+b+c+d$; and instead of $\{(a\times b)\div c\}\div d$, we write $\frac{ab}{cd}$; but if the multiplicand or the dividend is a sum or a difference, the bracket must not be omitted. Thus it will not do to write $a+b\times c$ instead of $(a+b)c$, nor $a-b\div c$ instead of $\frac{a-b}{c}$.

Whenever a positive or negative sign precedes a quantity within brackets, the sign affects the whole quantity. If the whole quantity is to be added, + precedes it, and the signs within the bracket remain unchanged; but if the whole quantity is to be subtracted, − precedes it, and the signs within the bracket must be changed. Thus $+(x+y-z)=x+y-z$, and $(x^2-2xy-y^2)=x^2-2xy-y^2$; but $-(x+y-z)=-x-y$

$+z$, and $-(x^2-2xy-y^2)=-x^2+2xy+y^2$. In the case of a double bracket, we have $3x-\{(x-3z)-(2y-z)\}=3x-(x-3z)+(2y-z)=3x-x+3z+2y-z=2x+2y+2z$.

The preceding remarks are equally applicable to fractions with numerators of more than one term, as often as a change of form is necessary. Thus $-\frac{x+y-z}{2}$, [or $-\frac{1}{2}(x+y-z)$]$=-\frac{x}{2}-\frac{y}{2}+\frac{z}{2}$, [or $-\frac{1}{2}x-\frac{1}{2}y+\frac{1}{2}z$]; and $-\frac{1}{4}(x-y)$, when multiplied by 4, becomes $-(x-y)$, or $-x+y$.

Exercise VIII.

Simplify as far as possible

1. $(3b^2+by)+(3a^2-by)-(b^2+a^2)$.
2. $(bcd-4x)+(2x+a^2y^2)+(2x-bcd-a^2y^2)$.
3. $(7x^2y+3ab^3)-(ab^3+3d^2+6x^2y)$.
4. $\{3d^2x-(5c-xy^3z)\}+(4c-8d^2x-xy^3z)$.
5. $(7ayz-p^2r)-(y^3-2ayz)+(y^3+2p^2r-5ayz)$.
6. $(mn^2-5a^2b^3)-(4y-6mn^2)-\{a^2b^3+(y-6a^2b^3)\}$.
7. $(4def+3ay^2z)-(9a^2x^2+2ay^2z)+(17def-ay^2z)$.
8. $(3b^4+8cyz)+\{(7-b^4)-(7-8b^4)\}-(8cyz+6b^4)$.
9. $(12b^2e^4+cx^3)-\{4m^2p-(4b^2e^4-6m^2p)\}+(8b^2e^4-2cx^3)$.
10. $(11a^2bc^4+3a^3b^2)-\{20b^3+(a^3b^2-4a^2c)\}-\{2a^3b^2+(4a^2c+3a^2bc^4)\}$.
11. $\{(3x^2y^2z^2-2yz^2)-(xyz-yz^2)\}-(2x^2y^2z^2-xyz)-(x^2y^2z^2-yz^2)$.
12. $\{5b^3z^5-(4p^2q^2+m^3n)\}-\{(20b^3z^5-1)-(6m^3n+16b^3z^5)\}-\{p^2q^2+(5m^3n-3p^2q^2)\}$.

It must be borne in mind that there are literal as well as numerical coefficients. $mx^2+nx^2-px^2+qx^2$ is an expression in which the several quantities m, n, p, and q, are coefficients of the common factor x^2, or *vice versâ*. This is very conveniently shown by means of a bracket, thus, $(m+n-p+q)x^2$; $ax^n+bx^n+cx^n+dx^n=(a+b+c+d)x^n$.

Exercise IX.

Collect coefficients in the following:

1. $ax+bx+3a^2x^2+4b^2x^2$.
2. $ax^3+bx^3+cx^3$.
3. $px^2+qx^2+ax+rx^2+bx$.
4. $mxy+by^2+nxy-pxy+dy^2$.
5. $ax+ay+by-cx+bx-cy$.
6. $xyz+az^3-xz^3+x^2z+x^2y^2+y^3z-xy^2$.
7. $acy+b^2x-a^3cx+c^2x-b^2cy$.
8. $px^2y^2-q^2xy^2+ax^2y^2-a^3xy^2+pxy^2$.
9. $mnx-nyx^2+xyz^2-(anx+by)+c^2y$.

10. $a^2bc^3x+(b^3c^2+ax^3-dx)-p^2x^3-c^2q$.

11. $4ab^2+b^2-3cyz+2bx-(dx+5a^2yz)$.

12. $ac(x+y)+3b(x+y)+2ab-4ax$.

13. $5be^3(x+1)-abe^3(x+1)+4a(x+1)$.

14. $6x^2y(p-q)-2xy^2-2bcy(p-q)+ay^2$.

15. $4a^2b^2c^2+12b^3c^2-20a^2b^2+12ab^3$.

16. $(mn+x)(y^2-z^2)-(3ab+ay)(y^2-z^2)$.

17. $(3a^2b+2abc^3)(y^3-1)+(abc^3-a^2b)(y^3-1)$.

18. $4(x-y)^3+3a(x+y)-(bc^2x+bc^2y)+abx^3-ab+3(x-y)^3$.

CHAPTER III.

MULTIPLICATION.

In simple algebraical quantities, multiply the numerical coefficients and letters, and to the product prefix the proper sign, which will be *plus* if both multiplier and multiplicand have the same sign, but *minus* if they have different signs.

Thus $6a\times 3b=18ab$; $-4a\times 5c=-20ac$; $3b\times -3c=-9bc$; $-7a\times -8b=+56ab$.

The product of two numbers is positive or negative, according as the signs of the factors are like or unlike. Like signs give +, unlike signs give −. For instance:

I. $+a\times +b=+ab$, because $a\times b$ signifies that $+a$ is to be taken b times, or added b times; the sum or the product must therefore be positive.

II. $-a\times +b=-ab$; for when $-a$ is multiplied by b, the meaning is that $-a$ is to be repeated b times, or that the quantity a is to be subtracted b times, or that a multiplied by b is to be subtracted: hence the sign $-ab$.

III. $-a\times -b=+ab$, since, considering $-a$ as an insulated negative quantity, or directly opposite in its nature to a positive quantity, to subtract it once is to reduce the $-a$ to $+a$, ∴ to subtract it b times would be $+ab$.

IV. $+a\times -b=-ab$, as in II. Where a is to be multiplied by $-b$, the signification is, that a is to be subtracted b times; hence the result must be $-ab$.

Or the reason for the rule may be thus shown. Suppose we have to multiply $m-n$ by $p-q$. Then $(m-n)(p-q)=(m-n)x$ [using x instead of $p-q$], which $=mx-nx$, or $m(p-q)-n(p-q)$, which $=(mp-mq)-(np-nq)$, or $mp-mq-np+nq$;

whence it is clear that $+m\times+p=mp$ or $+mp$,

$$\begin{aligned}&\text{and } +m\times-q=-mq,\\ &\text{and } -n\times+p=-np,\\ &\text{and } -n\times-q=+nq.\end{aligned}$$

Where the product of a number of simple quantities of different signs is sought, multiply all together, and the result will be positive or negative, according as the number of negative signs happens to be even or odd.

Thus, $-a\times-b\times+c=+abc$,

$$\begin{aligned}&-a\times-b\times-c=-abc,\\ &-a\times+b\times-c\times-d\times-e=+abcde,\\ &(-a)^2=+a^2;\ (-a)^3=-a^3;\ (-a)^4=+a^4;\ (-a)^5=-a^5;\end{aligned}$$

and generally, $(-a)^{2n}=+a^{2n}$; $(-a)^{2n+1}=-a^{2n+1}$.

It will be easily understood from the nature of indices or exponents, that the powers of quantities are multiplied by adding together their indices. For the product of a^4b^2c by a^3bc, we do not write $a^4a^3b^2bcc$, but add the exponents of a, b, and c, and obtain $a^7b^3c^2$. This is expressed by the general formula, $a^m\times a^n=a^{m+n}$, $b^m\times b^n\times b^p=b^{m+n+p}$, &c.; or in few words, to obtain the product of two or more powers of the same letter, write down the letter with an exponent equal to the sum of the exponents of the several factors.

Where the multiplier as well as the multiplicand is a compound quantity, all the terms of the latter must be multiplied successively by each term of the former, and the sum of the products will be the product of the two quantities required.

The multiplier is placed under the multiplicand, but on the left side.

N.B. The letters composing the factors of both multiplier and multiplicand should be arranged in the order of the alphabet previous to multiplication.

Ex. Let $-7ab+5b^2+9a^2$ be multiplied by $8ab$. Having first arranged the terms according to the descending powers of a, the multiplier is placed under the multiplicand, thus:

$$\begin{array}{l}9a^2-7ab+5b^2\\ 8ab\\ \hline 72a^3b-56a^2b^2+40ab^3.\end{array}$$

Then, $9a^2\times8ab=72a^3b$; next, $-7ab\times8ab=-56a^2b^2$; lastly, $8ab\times5b^2=40ab^3$. The separate products are summed up, and the whole obtained as above.

If the multinomial $-(1\frac{1}{2}a^3-\frac{1}{3}a^2+a-11)$ is to be multiplied by $\frac{1}{4}a^2$, we write,

$$\begin{array}{ll} & -1\frac{1}{2}a^3+\frac{1}{3}a^2-a+11, \text{ and multiplying}\\ \text{by} & \frac{1}{4}a^2\\ \hline \text{obtain} & -\frac{3}{8}a^5+\frac{1}{12}a^4-\frac{1}{4}a^3+2\frac{3}{4}a^2,\\ \text{or,} & -(\frac{3}{8}a^5-\frac{1}{12}a^4+\frac{1}{4}a^3-2\frac{3}{4}a^2).\end{array}$$

$$\begin{array}{ll}
\text{If} & -3x^4-5x^3+5\tfrac{1}{3}x^2-3\tfrac{1}{5}x+\tfrac{8}{9} \text{ be multiplied by} \\
 & -9\tfrac{1}{4}x^2 \\
\hline
\text{we obtain} & 27\tfrac{3}{4}x^6+46\tfrac{1}{4}x^5-49\tfrac{1}{3}x^4+29\tfrac{3}{5}x^3-8\tfrac{2}{9}x^2 \text{ product.}
\end{array}$$

Let $3a^3b-2a^2b^2+ab^3$ be multiplied by
$2a^2-ab-5b^2$, then

$$\begin{array}{lrrrrr}
(3a^3b-2a^2b^2+ab^3)\times 2a^2 = & 6a^5b & -4a^4b^2 & +2a^3b^3 & & \\
(3a^3b-2a^2b^2+ab^3)\times -ab = & & -3a^4b^2 & +2a^3b^3 & -a^2b^4 & \\
(3a^3b-2a^2b^2+ab^3)\times -5b^2 = & & & -15a^3b^3 & +10a^2b^4 & -5ab^5 \\
\cline{2-6}
\left.\begin{array}{r}\text{then adding these partial products,}\\ \text{we obtain as total}\end{array}\right\} & 6a^5b & -7a^4b^2 & -11a^3b^3 & +9a^2b^4 & -5ab^5
\end{array}$$

Again, let $a+b+c$ be multiplied by $b+a-c$. Arranging the multiplier in alphabetical order, we have

$$\begin{array}{lrrrrr}
\text{Multiplicand,} & a+b+c & & & & \\
\text{Multiplier,} & a+b-c & & & & \\
\cline{2-2}
 & a^2+ab+ac & & & & \\
 & & +ab & & +b^2+bc & \\
 & & & -ac & & -bc-c^2 \\
\cline{2-6}
\text{Product,}\ . & a^2 & +2ab & * & +b^2\ \ * & -c^2.
\end{array}$$

To find the product of $(z^3-2z^2+z-9)\times(8z-6+4z^2)$. Arranging the multiplier, we proceed as before :

$$\begin{array}{rrrrrr}
z^3 & -2z^2 & +z & -9 & & \\
4z^2 & +8z & -6 & & & \\
\hline
4z^5 & -8z^4 & +4z^3 & -36z^2 & & \\
 & +8z^4 & -16z^3 & +8z^2 & -72z & \\
 & & -6z^3 & +12z^2 & -6z & +54 \\
\hline
4z^5 & * & -18z^3 & -16z^2 & -78z & +54.
\end{array}$$

Let x^2-ax+b^2 be multiplied by

$$\begin{array}{l}
x-b \\
\hline
x^3-ax^2+b^2x \\
-bx^2+abx-b^3 \\
\hline
x^3-(a+b)x^2+b(a+b)x-b^3.
\end{array}$$

Let $(ba^2+3-ba-a^2+a)(a-b+ab^2-1+ab)$ be given ; then arranging the terms in the order of the powers of the letters a and b, we have, by bracketing,

$$\begin{array}{l}
(b-1)a^2-(b-1)a+3, \text{ multiplicand ; and} \\
(b^2+b+1)a-(b+1), \text{ multiplier.} \\
\hline
(b-1)(b^2+b+1)a^3-(b-1)(b^2+b+1)a^2+3(b^2+b+1)a \\
\qquad\qquad -(b-1)(b+1)a^2+(b-1)(b+1)a-3(b+1). \\
\hline
\end{array}$$

To obtain the sum of these partial products, it will be necessary to remove the brackets, by performing the operations they are meant to facilitate; then we find that

$$(b-1)(b^2+b+1)=b^3-1, \text{ and } (b-1)(b+1)=b^2-1;$$

$\therefore$ the coefficients of a^3, a^2, and a become, after addition,

$$(b^3-1),\ (-b^3+1-b^2+1), \text{ and } (3b^2+3b+3+b^2-1);$$

which, simplified, are

$$(b^3-1),\ -(b^3+b^2-2), \text{ and } (4b^2+3b+2).$$

The required product is, therefore,

$$(b^3-1)a^3-(b^3+b^2-2)a^2+(4b^2+3b+2)a-3(b+1).$$

EXERCISE X.

1. $(3a+4b)\times 9b$.
2. $(5b+d-3e)\times -6a$.
3. $(a^2+c^2-d^2)\times 3ab$.
4. $(a+b)\times(a+b)$.
5. $(a+b)\times(a-b)$.
6. $(a-b)\times(a-b)$.
7. $(a+b)^2\times -5a^2b^2$.
8. $(a-b)^2\times(a+b)^2$.
9. $(x^5+y^5)\times(x^5-y^5)$.
10. $(2x-3y)\times(-3x+4y)$.
11. $(2x^2-3xy-y^2)\times(x-2y)$.
12. $(3a-2b)^2\times(5a+7b)^2$.
13. $(x^4-3x^3+2x^2-x+5)\times(3x^3-2x^2-5x+3)$.
14. $(a^3+a^2b+ab^2+b^3)\times(3a^2-3ab+4b^2)$.
15. $(3a^4b^2-6a^3b^3+4a^2b^4+ab^5)\times(2a^2b-5ab^2+2b^3)$.
16. $(\frac{3}{4}x^3-\frac{2}{3}x^2y+\frac{1}{2}xy^2-\frac{1}{5}y^3)\times(\frac{2}{3}x^2+\frac{1}{2}xy-\frac{1}{4}y^2)$.
17. $(2x^4-3x^3y+2y^4-2xy^3+4x^2y^2)\times(2xy^2-y^3+\frac{1}{2}x^3-\frac{1}{3}x^2y)$.
18. $(0{\cdot}5m^4-0{\cdot}4m^3n+1{\cdot}2m^2n^2+0{\cdot}8mn^3-1{\cdot}4n^4)\times(0{\cdot}4m^2-0{\cdot}6mn-0{\cdot}8n^2)$.
19. $5(x-3)(x+5)(2x-7)$.
20. $(a^2-17)(a^2-a+5)(a^2-7a+1)$.
21. $(x+y+z)(x+y-z)(x-y+z)(-x+y+z)$.
22. $(2x-x^2+4x^3-10+7x^5)\times(2x^4-2x^2+3x^3+2+4x)$.
23. $(x+a)(x-b)$.
24. $(x-a)(x+b)$.
25. $(x-a)(x-b)(x+c)$.
26. $(x^2-ax-b^2)\times(x^2+px-q^2)$.
27. $(x^2-3ax+6a^2)\times(x^2+5bx+8b^2)$.
28. $(x+a)^3\times(x-a)^3$.
29. $(x^5+x^4y+x^3y^2+x^2y^3+xy^4+y^5)\times(x-y)$.
30. $(x^6-x^5y+x^4y^2-x^3y^3+x^2y^4-xy^5+y^6)\times(x+y)$.
31. $(x^p+y^r)\times(x^p-y^r)$.
32. $(3x^{4n+1}-4x^{3n}+2x^{2n-1}-x^{n-2})\times(2x^{4n+1}-5x^{3n}-2x^{2n-1}+x^{n-2})$.
33. $(x^{n-1}-x^{n-2}y+x^{n-3}y^2-x^{n-4}y^3+x^{n-5}y^4)\times(x+y)$.

Before proceeding to Division, the learner should exercise himself in resolving the products of compound quantities into their several factors. A great deal of attention is required for this, but much trouble will be afterwards saved by remembering the following results, many of which will have already been noticed in Ex. X, and which had now best be committed to memory.

I. $(a+b)(a-b)=a^2-b^2$; or, The product of the sum and difference of two numbers is equal to the difference of their squares.

II. $(a+b)^2=a^2+2ab+b^2$; that is, The square of the sum of two numbers is equal to the sum of their squares, plus twice their product.

III. $(a-b)^2=a^2-2ab+b^2$; that is, The square of the difference of two numbers is equal to the sum of their squares, minus twice their product.

IV. $(a+b)^2-(a-b)^2=4ab$; that is, The square of the sum of two numbers, minus the square of their difference, is equal to four times their product.

V. $(a+b)^2+(a-b)^2=2a^2+2b^2=2(a^2+b^2)$; that is, The square of the sum of two numbers, plus the square of their difference, is equal to twice the sum of their squares.

VI. $(a+b)^3=a^3+3a^2b+3ab^2+b^3$.

VII. $(a-b)^3=a^3-3a^2b+3ab^2-b^3$.

VIII. $(a-b)(a^2+ab+b^2)=a^3-b^3$.

IX. $(a+b)(a^2-ab+b^2)=a^3-b^3$.

X. $(a-x)(b-x)=ab-(a+b)x+x^2$.

In examples VI. and VII. another form might be given to the expression. The cube of $(a+b)=(a+b)(a^2+2ab+b^2)$, and the cube of $(a-b)$ $=(a-b)(a^2-2ab+b^2)$. In the forms given, however, it will be seen that as the powers of a decrease, those of b increase; and that the signs are all plus in Ex. VI., and alternately plus and minus in Ex. VII. This matter will be recurred to in the chapter on Involution.

We may now proceed to resolve a few products into their simplest forms.

1. Supposing $(4a^4-9b^6)$ is taken. Here we see the difference of two squares, the roots of which are $2a^2$ and $3b^3$; it is thus equal to the product (I.) of the sum and difference of these roots; so that $(4a^4-9b^6)=(2a^2+3b^3)(2a^2-3b^3)$.

2. In searching for the factors of x^2+9y^2-6xy, we observe that we have here the sum of two squares, minus twice the product of their roots; and the expression according to III. may be thus written, $(x-3y)(x-3y)$ $=(x-3y)^2$.

3. Let it be required to resolve $m^4-m^2n^2+m^3n-mn^3$ into its original factors. As each of the terms of the given form contains the factor m, we begin by writing $m(m^3-mn^2+m^2n-n^3)=A$. But $m^3-mn^2=m(m^2-n^2)$, and $m^2n-n^3=n(m^2-n^2)$, and substituting these for A, we have $m\{m\times(m^2-n^2)+n(m^2-n^2)\}$, or $m(m+n)(m^2-n^2)$; and because by I. m^2-n^2 $=(m+n)(m-n)$, therefore $A=m(m+n)(m+n)(m-n)$.

4. To resolve $a^2-b^2-c^2+2bc$ into its simplest factors. Let it be

thus expressed, $a^2-(b^2-2bc+c^2)$; then because $b^2-2bc+c^2=(b-c)^2$, it is by III. $a^2-(b-c)^2$, that is, according to I. $(a+b-c)(a-b+c)$.

5. Similarly $-a^3-b^3+a^2b+ab^2$ is thus written, $a^2b+ab^2-(a^3+b^3)$; then, according to IX., $a^3+b^3=(a+b)(a^2-ab+b^2)$, and $a^2b+ab^2=ab\times(a+b)$; therefore the whole $=ab(a+b)-(a^2-ab+b^2)(a+b)=A$. Now as both terms contain the factor $(a+b)$, we may write $(ab-a^2+ab-b^2)\times(a+b)$, or $(-a^2+2ab-b^2)(a+b)$. But $(-a^2+2ab-b^2)=-(a^2-2ab+b^2)=-(a-b)^2=-(a-b)(a-b)=(b-a)(a-b)$; wherefore A is reduced to $(b-a)(a-b)(a+b)$.

6. Resolve $\frac{1}{5}a^2+\frac{4}{5}ab+\frac{1}{2}ac-b^2-\frac{1}{2}bc$ into its simplest form. Instead of $\frac{4}{5}ab$ write $ab-\frac{1}{5}ab$; then the expression becomes $\frac{1}{5}a^2-\frac{1}{5}ab+ab-b^2+\frac{1}{2}ac-\frac{1}{2}bc=\frac{1}{5}a(a-b)+b(a-b)+\frac{1}{2}c(a-b)$, or $(\frac{1}{5}a+b+\frac{1}{2}c)(a-b)$.

7. Suppose we have to simplify the following :

$A=-x^4-y^4-z^4+2x^2y^2+2x^2z^2+2y^2z^2$. Attentive consideration will show that if instead of $2x^2y^2$ we write $4x^2y^2-2x^2y^2$, the expression will change into one that consists of the difference of two squares; for we have,

$A=4x^2y^2-x^4-2x^2y^2-y^4+2y^2z^2+2x^2z^2-z^4$.
$=4x^2y^2-(x^4+2x^2y^2+y^4-2y^2z^2-2x^2z^2+z^4)$.
$=4x^2y^2-\{(x^4+2x^2y^2+y^4)-2(x^2+y^2)z^2+z^4\}$, or see II.
$=4x^2y^2-\{(x^2+y^2)^2-2(x^2+y^2)z^2+z^4\}$, according to III. Thus
$A=4x^2y^2-\{(x^2+y^2)-z^2\}^2=4x^2y^2-(x^2+y^2-z^2)^2$, instead of which according to I. we may write $(2xy+x^2+y^2-z^2)(2xy-x^2-y^2+z^2)$. &c. &c.

EXERCISE XI.

Resolve into their factors

1. a^2b-ab^2.
2. $-a^2b-ab^2-ab$.
3. $15ac-10bc-3ad+2bd$.
4. $4a^4-4a^2b+b^2$.
5. $y^4-3y^3+3y^2-y$.
6. $x^3-xy^2-x^2z+y^2z$.
7. $x^2+13x+36$.
8. $x^2+29x+210$.
9. x^2+2x+1.
10. x^3+x-12.
11. $x^2-4x-45$.
12. $x^2-11x-180$.
13. $\frac{1}{4}x^4-\frac{1}{16}x^2$.
14. $9a^4+9a^3-18a^2$.
15. $x^2+10x+xy+8y+16$.
16. $a^2+2ab+b^2-c^2+2cd-d^2$.
17. $bm+bn-cm-cn+bxy-cxy$.
18. $8a^3b-27b^4$.
19. x^3-x.
20. $a^2b+ab^2+b^3-a^2c-abc-b^2c$.
21. $-x^3-y^3+x^2y+xy^2$.
22. $a^2-2ab+b^2-c^2$.
23. $2a^4+4a^3+2a^2+4a$.
24. $a^4+b^4+c^4-2a^2b^2-2a^2c^2-2b^2c^2$.

25. $6a^2+13ab-5b^2$.

26. $18a^2-8b^2$.

As a further exercise in multiplication, and the application of the preceding formulæ, it is required to simplify the following.

EXERCISE XII.

Simplify

1. $(x+8)^3-x(2x-3)^2+(9x^3-5x-33)$.
2. $(x^2-5x-7)(x+5)-3(x^2-13)(x+5)-(5x^3-13x-19)$.
3. $-2x(x^3-5x-9)+11\frac{1}{3}(x-7)^2(x-9)^2-(x^4-5x+35)$.
4. $5\{x^4-2x(x+9)\}-8\{x^5-7(x^2-3x+4)\}-(15x^3-11x-3)$.
5. $\frac{3}{4}\{(x-5)^2-(3x-11)^2\}^2-\frac{5}{6}(x^2-13x+8)^2-\frac{2}{3}(x^4+15x^2-17)$.
6. $(2x+a)^2(3x-b)^2-(x-2a)^2(x-3b)^2+(x^2-3ax-b^2)^2$.
7. $\frac{1}{2}\{(x+a)^4-(x-a)^4\}-\frac{2}{3}(x-a)^3(x+a)+4(2x^4-7x^3a+9x^2a^2-15xa^3+99a^4)$.
8. $(3a^2-\frac{1}{2}a+\frac{1}{3})^2-(7a^4+a^3-1\frac{1}{6}a^2-\frac{1}{2}a)$.
9. $\frac{2}{3}x(x^2-3x-5)^2-\frac{3}{4}(x-5)^2(2x+3)^2(-x+1)-2x(x-1)(x-2)(x+3)(x-4)$.
10. $(a+b-c)^2-(a+b)(a-c)-(a-b)(b-c)-(b-c)^2-b^2$.

CHAPTER IV.

DIVISION.

WHEN one of the two factors of a product is given together with the product, division enables us to find the other factor.

I. Coefficients are divided as common numbers, and the letters which are of like powers in both divisor and dividend do not appear in the quotient. Thus $15abcd\div5ab$ gives $3cd$; and $16ax^2yz\div2x^2y$ gives $8az$.

II. When like quantities have unlike indices or exponents, the index of the quotient is found by subtracting that of the divisor from that of the dividend. Thus $a^3\div a^2=a^1$ or a; $a^5\div a^3=a^2$.

Note. This is the place for explaining the meaning of powers with *negative exponents.* If a^3 is to be divided by a^7, the quotient is $a^3\div a^7=a^{3-7}=a^{-4}$. Supposing the expression takes a fractional form, $\frac{a^3}{a^7}$ or $\frac{1\times a^3}{a^4\times a^3}$, by leaving the common factor a^3 out of both divisor and dividend, we have

$$a^3\div a^7=\frac{a^3}{a^7}=\frac{1}{a^4};$$

$$\text{but } a^3 \div a^7 = a^{3-7} = a^{-4},$$
$$\text{therefore } a^{-4} = \frac{1}{a^4} = \frac{1}{a \times a \times a \times a}.$$

III. The quotient has the sign + when divisor and dividend are of like signs, and − when their signs are unlike. Ex.: $+16a \div +2a = 8$; $-20x \div -4x = +5$; $+16ab \div -2a = -8b$.

To prove this, let $\frac{a}{b} = p$, generally; then, because the quotient multiplied by the divisor is equal to the dividend, $b \times p = a$. But now,

$$+b \times +p = +a; \text{ therefore } \frac{+a}{+b} = +p.$$
$$-b \times +p = -a; \text{ therefore } \frac{-a}{-b} = +p.$$
$$-b \times -p = +a; \text{ therefore } \frac{+a}{-b} = -p.$$
$$+b \times -p = -a; \text{ therefore } \frac{-a}{+b} = -p.$$

Hence like signs give plus, and unlike signs give minus.

IV. If a quantity consisting of several terms is to be divided by a quantity of but one term, each term of the former is to be divided separately by the latter. Thus

$$\frac{a-b-c+d-e}{p} = \frac{a}{p} - \frac{b}{p} - \frac{c}{p} + \frac{d}{p} - \frac{e}{p}, \text{ and}$$
$$\frac{a-b-c+d-e}{-p} = -\frac{a}{p} + \frac{b}{p} + \frac{c}{p} - \frac{d}{p} + \frac{e}{p};$$

or, varying the form of expression, we have

$$\frac{a-b-c+d-e}{p} = (a-b-c+d-e) \times \frac{1}{p} = \frac{a}{p} - \frac{b}{p} - \frac{c}{p} + \frac{d}{p} - \frac{e}{p}, \text{ and}$$
$$\frac{a-b-c+d-e}{-p} = (a-b-c+d-e) \times -\frac{1}{p} = -\frac{a}{p} + \frac{b}{p} + \frac{c}{p} - \frac{d}{p} + \frac{e}{p}.$$

If the trinomial $5a^2b - 12ab^2 - 9b^2$ is divided by $3ab$, the partial quotients are $+1\frac{2}{3}a - 4b$ and $-\frac{3b}{a}$, which form the quantity $1\frac{2}{3}a - 4b - \frac{3b}{a}$. In the same way,

$$\frac{6a^4 - 7a^3 + 12a^2 - a}{4a} = 1\tfrac{1}{2}a^3 - 1\tfrac{3}{4}a^2 + 3a - \tfrac{1}{4};$$
$$\frac{12x^4y^2 - 7x^3y^3 + 9x^2y^4 - 11xy^5 + y^6}{3x^2y^2} = 4x^2 - 2\tfrac{1}{3}xy + 3y^2 - \frac{11y^3}{3x} + \frac{y^4}{3x^2};$$
$$\text{and } \frac{xy + xz + yz}{xyz} = \frac{1}{z} + \frac{1}{y} + \frac{1}{x}.$$

It will have been observed from the preceding example, that when the dividend, or any term of it, is not an exact multiple of the divisor, the quotient is expressed by a fraction, whose numerator and denominator are respectively those factors of the dividend and divisor which are prime to each other.

EXERCISE XIII.

1. Divide $4abc$ by $2ac$.
2. Divide $-27a^2bc$ by $-9ab$.
3. Divide $-15a^3b^2cd$ by $3ab^2d$.
4. Divide $121x^5y^3z$ by $11x^3y^2z$.
5. Divide $-65x^py^q$ by $-5x^ry^s$.
6. Divide $91a^pb^qc$ by $7a^rb^{2q}c^s$.
7. Divide $133a^5b^4c$ by $-19a^2b$.
8. Divide $-119a^5b^3cd$ by $17a^3b^3d$.
9. Divide $-117x^4y^3z$ by $-13x^2y$.
10. Divide $-35a^2b^3$ by $7a^3b^2$.
11. Divide $15a^2b^3c$ by $-10ab^4c^3$.
12. Divide $2a^7bc^5d$ by $-5a^6bc^4$.
13. Divide $-9216a^4b^3c^2de$ by $96a^2bcd$.
14. Divide $a^3(x+y)^7z$ by $5a^2(x+y)^4$.
15. Divide $39a^p$ by $13a^{p+3}$.
16. Divide $12a^pb^qc$ by $-3a^pb^{-q}c^r$.
17. Divide $-x^5y^7$ by $9x^8y^{12}$.
18. Divide $(a+b)^x$ by $(a+b)^y$.
19. Divide $7a(x-y)^5$ by $21a(x-y)^4$.
20. Write the following fractions in the form of whole numbers:

$$\frac{a}{b},\quad \frac{2a}{5b},\quad \frac{27a^3b}{9a^2b^3},\quad \frac{a^pb^q}{a^{p+3}b^r},\quad \frac{a(x+y)}{b(x-y)}.$$

V. When both divisor and dividend are compound quantities, the common arithmetical process must be pursued, the student remembering to arrange the terms of both divisor and dividend according to the ascending or descending powers of some letter common to both.

EXAMPLE I.

$$a^3+a^2b+ab^2+b^3)3a^5 \qquad +4a^3b^2+4a^2b^3+ab^4+4b^5(3a^2-3ab+4b^2$$

$$\underline{3a^5+3a^4b+3a^3b^2+3a^2b^3}$$

First remainder $\quad -3a^4b+a^3b^2+a^2b^3+ab^4$

$$\underline{-3a^4b-3a^3b^2-3a^2b^3-3ab^4}$$

Second remainder $\quad 4a^3b^2+4a^2b^3+4ab^4+4b^5$

$$\underline{4a^3b^2+4a^2b^3+4ab^4+4b^5}$$

Third remainder $\quad 0$

The first term $3a^5$ of the dividend, divided by the first term of the divisor a^3, gives the first term $3a^2$ of the quotient. Then the terms of the divisor are multiplied by this $3a^2$, and like terms placed under like, that the product may be subtracted from the dividend. Dividing as before for the second term of the quotient, we get $-3a^4b\div a^3=-3ab$. Next, the divisor is multiplied by $-3ab$; and the product, deducted from the first remainder, leaves the second remainder $4a^3b^2+4a^2b^3+4ab^4+4b^5$, including the fresh term brought down. Dividing again for the third term of the quotient, we have $4a^3b^2\div a^3=4b^2$, and subtracting from the second remainder the product of this term and the divisor, we have 0 remainder. The division is thus finished, and the quotient found to be $3a^2-3ab+4b^2$.

It will be seen that a break was left in the dividend, because in the descending powers of a, a term was wanting; such a break was not indispensable, but it made the operation easier and more methodical.

Ex. 2. Let a^4-b^4 be divided by $a-b$.

$$\begin{array}{l}
a-b)\,a^4-b^4\,(a^3+a^2b+ab^2+b^3 \\
\qquad\;\; \underline{a^4-a^3b} \\
\qquad\qquad\; a^3b-b^4 \\
\qquad\qquad\; \underline{a^3b-a^2b^2} \\
\qquad\qquad\qquad\; a^2b^2-b^4 \\
\qquad\qquad\qquad\; \underline{a^2b^2-ab^3} \\
\qquad\qquad\qquad\qquad\; ab^3-b^4 \\
\qquad\qquad\qquad\qquad\; \underline{ab^3-b^4} \\
\qquad\qquad\qquad\qquad\qquad\; 0
\end{array}$$

Proof:

$$\begin{array}{ll}
a^3+a^2b+ab^2+b^3 & \text{quotient,} \\
\underline{a-b\qquad\qquad\quad} & \text{divisor,} \\
a^4+a^3b+a^2b^2+ab^3 & \\
\underline{\quad -a^3b-a^2b^2-ab^3-b^4} & \\
a^4\;\;\cdot\quad\cdot\quad\cdot\;-b^4 & \text{dividend.}
\end{array}$$

While applying the rules in (VI.), it may happen that in the arrangement of divisor and dividend, several terms may be found containing the common letter raised to the same power; these terms must be united by placing their coefficients between brackets, and the bracketing must be repeated as far as necessary.

For instance, let $b^3a^2-b^4-ab^3+a^3b^2+2a^2b+3ab^2-a^3b-2a^2$ be divided by a^2b-b^2+2a. Then arranging divisor and dividend according to the descending powers of a, we have for the dividend $b^2a^3-ba^3+b^3a^2+2ba^2-2a^2+3b^2a-b^3a-b^4$, and for the divisor ba^2+2a-b^2. Now bracketing, we have,

$$\begin{array}{rl}
 & ba^2+2a-b^2\big)\,b(b-1)a^3+(b^3+2b-2)a^2+(3b^2-b^3)a-b^4\big(\,(b-1)a+b^2 \\
 & \underline{\qquad b(b-1)a^3+2(b-1)a^2-b^2(b-1)a} \\
\text{1st remainder} & (b^3+2b-2-2b+2)a^2+(3b^2-b^3+b^3-b^2)a-b^4 \\
\text{which simplified becomes} & b^3a^2+2b^2a-b^4 \\
 & \underline{b^3a^2+2b^2a-b^4} \\
\text{second remainder} & \qquad 0
\end{array}$$

Ex. We have divided the first term of the dividend $b(b-1)a^3$ by the first term ba^2 of the divisor, which gives $\frac{b(b-1)a^3}{ba^2}=(b-1)a$, for the first term of the quotient. Next we multiply this first term of the quotient into the divisor, and subtract the product from the dividend. Then simplifying the remainder, and dividing, we get b^2 as the second term of the quotient, and subtracting the product of b^2 and the divisor, obtain 0 remainder; and $(b-1)a+b^2$ is the quotient required.

If the coefficients of the common letter are themselves compound, they may necessitate compound terms in the quotient. We shall take as an illustration of this the following example.

$$\begin{array}{rl}
 & (b-1)a^2-(b-1)a+3\big)\,(b^3-1)a^3-(b^3+b^2-2)a^2+(4b^2+3b+2)a-3(b+1)\big(\,(b^2+b+1)a-(b+1) \\
 & \underline{\qquad (b^3-1)a^3-(b^3-1)a^2+3(b^2+b+1)a} \\
\text{First remainder} & -(b^2-1)a^2+(b^2-1)a-3(b+1) \\
 & \underline{-(b^2-1)a^2+(b^2-1)a-3(b+1)} \\
\text{Second remainder} & \qquad 0
\end{array}$$

To obtain the first term of the quotient, $(b^3-1)a^3$ has been divided by $(b-1)a^2$, which gives $(b^2+b+1)a$. Subtract now the product of this term and the divisor, and we get for the coefficient of a^2, $-b^3-b^2+2+b^3-1$, or $-b^2+1$, or $-(b^2-1)$; and for that of a, $4b^2+3b+2-3b^2-3b-3$, or (b^2-1). The first remainder is thus $-(b^2-1)a^2+(b^2-1)a-3(b+1)$. Now divide $-(b^2-1)a^2$ by $+(b-1)a^2$, and that gives $-(b+1)$ for the second term of the quotient. Subtract from the first remainder the product of this second term of the quotient and the divisor, and 0 remains; so that the quotient is $(b^2+b+1)a-(b+1)$.

Where the divisor is not a factor of the dividend, a remainder will of course be left; it must be appended in fractional form to the quotient and, as in common arithmetic, added to the product of the two factors in the proof to give a proper result.

In dividing $a^4+a^3b-3b^2$ by a^2-b, the quotient is found to be a^2+ab+b, and the remainder ab^2-2b^2. The expression is thus written:

$$\frac{a^4+a^3b-3b^2}{a^2-b}=a^2+ab+b+\frac{(a-2)b^2}{a^2-b}$$

By simplifying, and by attention to the formulæ (p. 17), the division of compound quantities may be often performed by mere inspection. From I., VIII., and IX. (p. 17), may be easily deduced the following:

1. That the difference of two even like powers is divisible by both the sum and the difference of their roots.

$$\frac{a^2-b^2}{a-b}=a+b\,;\ \frac{a^2-b^2}{a+b}=a-b\,;\ \frac{a^4-b^4}{a-b}=a^3+a^2b+ab^2+b^3;\ \&c.$$

2. That the sum of two odd like powers is divisible by the sum of their roots, and their difference by the difference of their roots.

$$\frac{a^3+b^3}{a+b}=a^2-ab+b^2\,;\ \frac{a^5-b^5}{a-b}=a^4+a^3b+a^2b^2+ab^3+b^4;\ \&c.$$

The student will not fail to notice the order in which these several results come out, viz. in the descending powers of a and the ascending powers of b; and further, that the signs are all +, except when both terms of the divisor are +, and then the result is alternately + and −.

$$\frac{a^2b(b^2-c^2)}{ab(b-c)}=\frac{aab(b-c)(b+c)}{ab(b-c)}=a(b+c).$$

So also

$$\frac{4a^5-12a^4b+12a^3b^2-4a^2b^3}{2a^3-4a^2b+2ab^2}=\frac{4a^2(a-b)^3}{2a\,(a-b)^2}=2a(a-b).$$

The work may also be materially abridged when from the dividend we can throw out a factor which forms no part of the divisor. In such a case, the quotient is as many times too small as there are units in the factor thrown out; but a multiplication of the quotient by the factor thrown out restores the loss.

If $25\,a^4x-75\,a^4y-15\,a^3x+45\,a^3y-6\,ax+18\,ay-4\,x+12\,y$ is to be divided by $5a^2-3a-2$, the dividend may be written

$$25a^4(x-3y)-15a^3(x-3y)-6a(x-3y)-4(x-3y),$$

for which we may also write

$$(25a^4-15a^3-6a-4)(x-3y);$$

throw out $(x-3y)$, and then dividing $25a^4-15a^3-6a-4$ by $5a^2-3a-2$, we have $5a^2+2$ as quotient. Multiply it by $x-3y$, which gives $(5a^2+2)(x-3y)$, or $5a^2x-15a^2y+2x-6y$, the true quotient.

EXERCISE XIV.

1. Divide $6a^2-2ab+b^2$ by $2a$.
2. Divide $5a^3b-7a^2b^2+9ab^3$ by $-5ab$.
3. Divide $-35x^3+15x^2y-xy^2-y^3$ by $-5x^2$.
4. Divide $-(6x^4y^3-42x^3y^4+9x^2y^5+16xy^6)$ by $6x^2y^3$.
5. Divide $a^2+b^2+c^2$ by $2a^2$.
6. Divide $2a^2(x+y)^4-2ab(x+y)^3+xy$ by $-4a(x+y)^2$.
7. Divide $-7ab(x^2-y^2)-8a^2(x-y)^2+9b^2(x-y)$ by $-12ab(x-y)$.
8. Divide $25c^2d^3e+40c^4d^5e^2f$ by $5d^2e+8c^2d^4e^2f$.
9. Divide $-73a^2-25+56a^4+95a-59a^3$ by $-11a+7a^3-3a^2+5$.
10. Divide $6a^3-13a^2b+4ab^2+3b^3$ by $2a-3b$.

11. Divide $26a^3b^3+a^6+6b^6-17ab^5-5a^5b-2a^4b^2-a^2b^4$ by a^2-3b^2-2ab.

12. Divide x^5-y^5 by $x-y$.

13. Divide x^5+y^5 by $x+y$.

14. Divide $9a^2-12ab+4b^2-x^2$ by $3a-2b+x$.

15. Divide $4x^3+4x^2-29x+21$ by $2x-3$.

16. Divide x^4-y^4 by $x^3+x^2y+xy^2+y^3$.

17. Divide $x^4y^2-y^6+x^6-x^2y^4$ by $x^3+x^2y+y^3+xy^2$.

18. Divide $35a^2+4a+18a^3+45a^4-4$ by $7a+9a^3-2$.

19. Divide $26ab^4-17a^3b^2+8a^5-22a^4b-8b^5+48a^2b^3$ by $2a^2-4b^2-3ab$.

20. Divide $a^2b^2c+3a^2b^3-3abc^3+bc^4-a^2c^3+b^5-2b^3c^2+3ab^3c-3a^2bc^2$ by b^2-c^2.

21. Divide $a^4-4a^3c-4a^2b^2+4a^2c^2+4ab^2c-b^2c^2$ by $a^2-2ab-2ac+bc$.

22. Divide $-a^4b^2+b^2c^4-a^2c^4-a^6+2a^4c^2+b^6+2b^4c^2+a^2b^4$ by $a^2-b^2-c^2$.

23. Divide $6x^{8n+2}-23x^{7n+1}+18x^{6n}-x^{5n-1}-3x^{4n-2}+4x^{3n-3}-x^{2n-4}$ by $2x^{4n+1}-5x^{3n}-2x^{2n-1}+x^{n-2}$.

24. Divide $5m^7-22m^6n+12m^5n^2-6m^4n^3-4m^3n^4+8m^2n^5$ by $m^3-4m^2n+2n^3$.

25. Divide $12c^4-27c^2d^2+18cd^3-3d^4$ by $6c^2+3d^2-9cd$.

26. Divide $(a^6-x^6)(a+b)$ by $(a^2-x^2)(a+b)^2$.

27. Divide $a^3+a^2b+2b^3$ by a^2+b^2.

28. Divide $\frac{2}{7}a^5-a$ by $\frac{1}{2}a^2+1$.

29. Divide $8x^{-6}y^2a^{-5}-8x^{-5}y^3a^{-6}-6x^{-4}y^4a^{-7}$ by $2x^{-4}ya^{-2}-3x^{-3}y^2a^{-3}$.

30. Divide $15x^{-7}y^3-3x^{-6}y^4-11x^{-5}y^5+17x^{-4}y^6-6x^{-3}y^7$ by $5x^{-3}y+4x^{-2}y^2-3x^{-1}y^3$.

31. Divide $1\frac{1}{5}x^5+\frac{11}{144}a^4x+\frac{299}{480}a^2x^3-\frac{7}{8}ax^4-1\frac{1}{2}a^3x^2+\frac{5}{9}a^5$ by $\frac{5}{8}ax-1\frac{1}{2}x^2+\frac{2}{3}a^2$.

32. Divide $0{\cdot}48x^4y-1{\cdot}2y^5+2{\cdot}49xy^4+0{\cdot}48x^5-1{\cdot}8x^3y^2$ by $0{\cdot}4x^3-0{\cdot}2x^2y-0{\cdot}7xy^2+0{\cdot}8y^3$.

33. Divide $20(x+y)^5-46(x+y)^4z+84(x+y)^3z^2-78(x+y)^2z^3+64\times(x+y)z^4-32z^5$ by $5(x+y)^3-4(x+y)^2z+5(x+y)z^2-4z^3$.

In this and the two following examples, a may be substituted for $x+y$ as a simplification of the division; then, naturally, $(x+y)^2=a^2$, $(x+y)^3=a^3$, &c.; but then in the quotient, the original value $(x+y)$ of a must be restored.

34. Divide $2(x+y)^5-6(x+y)^4z-5(x+y)^3z^2+8(x+y)^2z^3-4(x+y)z^4+2z^5$ by $2(x+y)^3-3(x+y)^2z+4(x+y)z^2-5z^3$.

35. Divide $8(x+y)^{-8}z^{-3}-16(x+y)^{-7}z^{-4}-28(x+y)^{-6}z^{-5}+9\times(x+y)^{-5}z^{-6}+9(x+y)^{-4}z^{-7}$ by $4(x+y)^{-5}z^{-2}+2(x+y)^{-4}z^{-3}-3(x+y)^{-3}z^{-4}$

36. Divide $5a^6+12a^4b^2-12a^2b^4-3b^6+8ab^5-3a^3b^3-4a^5b$ by $4a^3-2a^2b+3ab^2-4b^3$.

37. Divide $x^2y^2+3x^4-2y^4+4xy^2z+5x^3y+5xy^3-10x^2yz-6x^3z$ by $3x^2-2y^2+5xy$.

38. Divide $8ax^5-16c^3x^3+10ad^4x+8ac^3x^2+12b^2x^4-20d^4x^2-15abd^4-4a^2x^4+9ab^3x^2-12abc^3x+6a^2bx^3-6ab^2x^3$ by $2ax^3+3b^2x^2-4c^3x-5d^4$.

39. Divide $12a^5-31a^4b+(20b^2-29c^2)a^3+(37bc^2+18c^3)a^2+(15c^4-24bc^3)a-30c^5$ by $3a^2-4ab-5c^2$.

40. Divide $ba^3+b^3a^2+b^3(b-1)a-(b-1)c-b(b-1)ca^{-1}$ by $(b-1)a+b(b-1)$.

CHAPTER V.

EQUATIONS.

Two symbolical expressions connected by the sign of equality, constitute an *equation*. The symbolical expressions themselves are called the sides of the equation. The several parts on both sides having the sign + or − between them are called the *terms* of the equation.

When merely two different forms of one and the same quantity are joined together, as for instance

$$(a\pm b)^2=a^2\pm 2ab+b^2$$

$$(a+b)(a-b)=a^2-b^2$$

the equation is said to be an *identity*.

Two different quantities may, however, be equated with a view to the determination of the value of some unknown quantity (x, y, z, &c.) by means of others that are known (a, b, c, &c.).

Any quantity may be considered the unknown.

So to determine the unknown quantity in an equation, that the equation shall be complete, or *satisfied*, is to *solve* the equation.

The equation $a+bx=c$ is solved by the value of $x=\left(\frac{c-a}{b}\right)$, because, as is seen at once,

$$a+b\left(\frac{c-a}{b}\right)=c \text{ is an identical equation.}$$

Equations are classed according to their dimensions, that is, according to the degree of expansion of the powers of the unknown quantities. In general, an equation is said to be of the n^{th} degree, when the highest exponent of the power of the unknown quantity is n.

$$a+bx=c$$

$$a+bx+cx^2=d$$

$$a+bx+cx^2+dx^3=e$$

$$a+bx+cx^2+dx^3+ex^4=f$$

are examples of equations of the first, second, third, and fourth degrees. The last three are also called quadratic, cubic, and biquadratic equations.

In every equation, any term may be transposed, with a changed sign, to the opposite side, without altering the value of the equation.

For let the given equation be $(A)\pm G=(B)$, in which (A) denotes the aggregate of all the terms on the left side, excepting the single term G, and (B) denotes the aggregate of all the terms on the right side.

$$\text{Then } (A)\pm G\mp G=(B)\mp G,$$
$$\text{that is, } (A)=(B)\mp G.$$

If all the terms of an equation are multiplied or divided by one and the same number, the value of the equation remains unaltered.

As this is only an application of principles previously explained, no proof is needed. It must be remarked, however, that by means of the above, the terms of an equation may be cleared of fractions if both sides are multiplied by the L. C. M. of all the denominators. An equation may very often be simplified by dividing all its terms by some one quantity.

If all the terms of the equation

$$1+\tfrac{2}{3}x+\tfrac{3}{4}x=-\tfrac{5}{6}x+\tfrac{1}{2}x$$

are multiplied by 12, the least common multiple of the denominators 2, 3, 4, 6, we obtain

$$12+8x+9x=-10x+6x,$$

clear of fractions.

The equation $64-32x+16x=96x$, when divided by 16 in all its terms, becomes

$$4-2x+x=6x,$$

a simpler expression. The coefficients of an equation may always be divided by their G. C. M.

To solve any equation of the first degree containing one unknown quantity.

Rule I. Clear the equation of fractions if necessary.

II. Simplify the equation as far as possible.

III. By transposition of the terms, bring all known quantities to one side, and all unknown quantities to the other side of the equation; then by addition reduce the several terms to a single term on each side.

IV. Now divide both sides of the equation by the coefficients of the unknown quantity, the value of which will thus be found.

Ex. I. Let the given equation be

$$x+\frac{6x-8}{7}-\frac{3x-3}{5}=\frac{4x-4}{5}+\frac{20-x}{2}-4.$$

Multiplying by 70, we have

$$70x+60x-80-42x+42=56x-56+700-35x-280.$$

By transposition of the several terms we have

$$70x+60x-42x-56x+35x=80-42-56+700-280,$$

and by addition of the several terms on both sides,

$$67x=402;$$

then dividing both sides by 67, we find $x=6$.

Ex. II. The solution of the following equation,

$$ax-\frac{bx-b}{2a}+\frac{b-ab}{a}=\frac{2a^2+b}{2a}+\frac{abx+b}{a}-\frac{bx}{2a},$$

may thus be seen step by step.

$$2a^2x-bx+b+2b-2ab=2a^2+b+2abx+2b-bx,$$
$$2a^2x-bx-2abx+bx=2a^2+b+2b-b-2b+2ab,$$
$$2a^2x-2abx=2a^2+2ab,$$
$$ax-bx=a+b;$$
$$(a-b)x=a+b,$$
$$x=\frac{a+b}{a-b}.$$

A few more examples are appended to familiarise the student with the method of solution, &c.

Ex. III. To find the value of x in the equation $4x-3=2x+5$.

by transposition, $4x-2x=5+3,$

$\therefore \quad 2x=8,$

dividing by 2, $x=\frac{8}{2}=4.$

This is proved by substituting 4, the value of x, for x; thus $(4\times 4)-3=(2\times 4)+5$, or $13=13$.

Ex. IV. To find the value of x in the equation

$$\frac{5x}{12}-\frac{4x}{3}-13=\frac{7}{8}-\frac{13x}{6}.$$

clearing of fractions, $10x-32x-312=21-52x,$

transposing, . $10x-32x+52x=21+312,$

simplifying, . . . $30x=333,$

dividing by 30, ... $x=\frac{333}{30}=11\tfrac{1}{10};$

which may be proved as before.

Ex. V. What is the value of x in $(3a-x)(a-b)+2ax=4b(x+a)$?

multiplying, $3a^2-ax-3ab+bx+2ax=4bx+4ab;$

transposing and simplifying, . $ax-3bx=7ab-3a^2;$

or because $ax-3bx=(a-3b)x$, and $7ab-3a^2=a(7b-3a)$,

$$\therefore \quad (a-3b)x=a(7b-3a);$$

dividing, $x=\dfrac{a(7b-3a)}{a-3b}.$

Ex. VI. Find the value of x in the equation $\sqrt{(x^2+ax+b)}=x+p$.

squaring both sides, $x^2+ax+b=x^2+2px+p^2$,

transposing, $x^2-x^2+ax-2px=p^2-b$,

or $(a-2p)x=p^2-b$;

$$\therefore\ x=\frac{p^2-b}{a-2p}.$$

Ex. VII. Find the value of x in the equation

$$79-2\times\frac{18-\frac{4}{5}x}{3}-11\times\frac{34+\frac{1}{3}x}{13}-3(x-1)=0.$$

The successive stages of the operation are as follow:

$$3081-26(18-\tfrac{4}{5}x)-33(34+\tfrac{1}{3}x)-117(x-1)=0,$$

$$3081-468+\frac{104}{5}x-1122-11x-117x+117=0,$$

$$\frac{104}{5}x-11x-117x=-3081+468+1122-117,$$

$$\frac{104}{5}x-128x=-1608,$$

$$104x-640x=-8040,$$

that is, $-536x=-8040$,

or $x=15$.

Exercise XV.

Required the value of x in the following equations:

1. $6-4x=21-7x$.
2. $5a-2x=7a+8b-4x$.
3. $ax\pm b=c$.
4. $ax-3b-2cx=x-4d$.
5. $9623-11x=17x+1223$.
6. $ax+3ab-4cd=6ab-4cd+bx$.
7. $13\frac{3}{4}-\frac{x}{2}=2x-8\frac{3}{4}$.
8. $2x+7+\frac{3}{2}x=6x-23$.
9. $12\frac{1}{4}+3x-6-\frac{7x}{3}=\frac{3}{4}x-5\frac{3}{8}$.
10. $\dfrac{15}{2x+3}=\dfrac{19}{4x-5}$.
11. $\dfrac{5x-90}{3}=\dfrac{160-2x}{5}$.
12. $\dfrac{ax-b^2}{a}=\dfrac{ax+b^2}{b}$.
13. $\dfrac{cx}{2c-3x}-d=0$.
14. $5(8-x)-3(60-5x)-2(4-x)+40=0$.
15. $7(x-16)-9(37-x)+2(111-3x)=3x+8$.
16. $a(a-x)=7b(x+a)+(a+b)(x-2a)$.
17. $\frac{6}{13}(x-1)+39=\frac{3}{4}(x+21)$.
18. $\frac{1}{4}(x+3)-\frac{1}{5}(63-x)+\frac{1}{3}(x+5)=0$.
19. $9\frac{3}{7}x-6\frac{2}{5}(1\frac{1}{2}x-3\frac{3}{5})+1000(6-\frac{1}{7}x)=0$.
20. $3(33-\frac{4}{7}x)-5(8-\frac{1}{8}x)+2(13-\frac{3}{14}x)=0$.
21. $\dfrac{qx}{p+q}-\dfrac{px}{p-q}+r=0$.
22. $\dfrac{5a+2x}{2x}-6=\dfrac{5}{2x}$.
23. $\dfrac{3x}{5}-\dfrac{7x}{10}+\dfrac{3x}{4}-\dfrac{7x}{8}=-15$.
24. $\dfrac{b^2x}{cd}-\dfrac{a^2}{b}+cx=\dfrac{ex}{d}-c+(a+c)x$.

25. $3{\cdot}25x-5{\cdot}007-x=0{\cdot}2-0{\cdot}34x.$

26. $\frac{7{\cdot}53x}{18}+100=\frac{2x}{5}+3{\cdot}86-\frac{x}{6}.$

27. $x=a+\frac{bc}{d}+\frac{cfx}{de}.$

28. $\frac{ex}{f}+\frac{cx}{d}+\frac{ax}{b}-g=h.$

29. $\frac{x}{a}-1-\frac{dx}{c}+3ab=0.$

30. $\frac{ab}{x}=bc+d+\frac{1}{x}.$

31. $\frac{a}{2x-3}-\frac{b}{3x-4}=0.$

32. $\frac{3a+x}{x}-5=\frac{6}{x}.$

33. $\frac{a^2x}{b-c}+cd=bx-ac.$

34. $\frac{a(d^2+x^2)}{dx}=ac+\frac{ax}{d}.$

35. $\frac{(c-d)^2}{a+b}+x=\frac{c-d}{a+b}x+a+b.$

36. $\frac{a}{bx}+\frac{c}{dx}+\frac{e}{fx}+\frac{g}{hx}=k.$

37. $a-\frac{b+c}{x}-(a-c)\times\frac{c^2}{a}=0.$

38. $\frac{ax-b^2}{x+b}-7a+3b=0.$

39. $\frac{a^3}{a^2-b^2}+ax-2a+bx-b\times\frac{a^2+ab-b^2}{a^2-b^2}=0.$

40. $\frac{a+3x}{4a}-\frac{7a-5x}{6b}+3-\frac{9x}{4}=\frac{x}{ab}+\frac{5x}{6b}.$

Practical application of Simple Equations.

Ex. 1. A wholesale brewer has 20 casks of beer at 8*s.* a cask; how many casks at 3*s.* each must he add to the first, that the mixture may be worth 5*s.* a cask?

SOLUTION.

Let us suppose that he must add x casks at 3*s.*, then the worth of the beer added is $3x$ shillings, while the value of the 20 given casks is $20\times8=160$*s.*; the worth of the mixed beer thus becomes $(160+3x)$ shillings, and it consists of $(20+x)$ casks; and as it must be at 5*s.* the cask, the whole worth of the mixture is expressed by $(20+x)5$*s.*

We have thus the equation:

$$\begin{aligned} (20+x)5&=160+3x,\\ \text{or }\ 100+5x&=160+3x,\\ \textit{i.e. }\ 5x-3x&=160-100,\\ 2x&=60,\\ x&=30, \text{ the number of casks at 3}\textit{s}. \end{aligned}$$

Proof. If 20 casks at 8*s.*, which are worth 160*s.*, are mixed with 30 casks at 3*s.*, which are worth 90*s.*, 50 casks are obtained worth 250*s.*; so that the price of 1 cask of this mixture is 250÷50 or 5*s.*

Ex. 2. A father being asked how old his son was, answered: "I am 40 years older than my son, but in 2 more years I shall be six times as old as he." What is the present age of the son?

SOLUTION.

Let $x=$ the present age of the son, then in 2 years his age will be $x+2$. The father will be then, because he is 40 years older than the son,

$x+42$, or also because he is 6 times as old as the son, $6(x+2)$. Thus we have

$$6(x+2)=x+42\,;$$

whence we find $6x+12=x+42$,

$$5x=30,$$

$x=6$; thus the son is 6 years old.

Ex. 3. A gentleman engaged with his servant for 9*l.* a year, and a suit of clothes, but after 5 months the servant left, receiving 2*l.* only and the suit of clothes; what was the worth of the latter?

Let x£ = the worth of the clothes,

Then wages for 12 months $=(9+x)$£,

and wages for 1 month $=\frac{9+x}{12}$,

$\therefore$ wages for 5 months $=\left(\frac{9+x}{12}\right)5=\frac{45+5x}{12}$;

but the servant was paid for 5 months only $2+x$£,

$$\therefore \frac{45+5x}{12}=2+x;$$

clearing of fractions, $45+5x=24+12x$,

$\therefore$ $x=3$£ = worth of the clothes.

Exercise XVI.

1. A certain number is doubled, and then multiplied by 4; further, 15 is added to make up 135. What is it?

2. From a piece of linen a dealer cut off $\frac{1}{4}$ and $\frac{1}{5}$, and found that they were together 27 yards. What was the length of the web?

3. From another piece he sold $\frac{1}{3}$ and $\frac{1}{5}$, and had then 21 yards remaining. What was its length?

4. Out of a purse $\frac{1}{5}$, $\frac{1}{6}$, a ten-pound note, and six sovereigns were taken, yet the purse was only half emptied. What did it contain?

5. One day $\frac{1}{8}$ of the pupils in a school were absent, and on the following day $\frac{1}{5}$ of the remainder; it was observed that on the first day six more were present than on the second. What was the number of the pupils?

6. An old man being asked his age, answered: "If I live 9 years longer, and can add to the age then attained one fourth of my years, I shall be 100." How old was he?

7. If from a certain number 18 are subtracted, the remainder multiplied by 11, and to the product 6 added, three times the original number will be obtained. What is it?

8. A number of apples is to be equally divided between two classes of 7 boys each, so that each class shall have $\frac{1}{7}$ of the whole and 10 more. How many will each boy have?

9. At 12 o'clock the hands of a watch are together; how often will

they come together again, and at what intervals, during the next twelve hours?

10. A labouring man agreed to work 8 months for 10*l.* and a fat pig. By an accident he was hurt at the end of the sixth month, and his wife claimed the pig and 5*l.* as his due. What did the pig weigh, supposing it worth 6*d.* a pound?

11. A man brought a quantity of eggs to market, and sold one half at 3 for five farthings, the other half at 4 for seven farthings; but in counting over the last he broke 20, so that each half sold for the same money. How many were there?

12. A web of linen 75 yards long was cut into two parts, so that one fourth of the largest piece contained 3 yards more than the fifth part of the other. How much was one piece longer than the other?

13. On a day just 12 hours long, a man asked: What o'clock is it? and received as answer—Half of the time that has passed is just equal to one third of the time remaining. What o'clock was it?

14. The number 451 must be divided into two parts, so that 14 added to the first shall equal the last when divided by the same number.

15. Two boys were talking about their marbles, when one said: "Sell me a dozen of yours, then we shall have an equal number." "No," replied his comrade; "I would rather buy as many of you, then I shall have three times your number." What were their numbers united?

16. Divide the number 237 into two parts, so that the one shall be contained $1\frac{1}{4}$ times in the other.

17. £1170 is to be divided among three persons in proportion to their ages. B is $\frac{1}{3}$ older than A, and C twice as old. What will each have?

18. Divide the number a into three parts, so that the first is proportioned to the second as m to n, and the second to the third as p to q.

19. A company of 90 persons consists of men, women, and children; the men are 4 times as many as the women, and there are 10 more children than grown-up people. What is the number of each?

20. Find three numbers such that the second divided by the first gives 2 in the quotient and 1 in the remainder; on the contrary, the third divided by the second gives 3 in the quotient and 3 in the remainder: the sum of the three numbers must be 70.

21. In a garden there were apple, pear, and cherry trees, currant and gooseberry bushes; in all 51. The trees were 5 more than the bushes; the cherry-trees 3 less than the apple-trees, and 2 more than the pear-trees; the currants 7 less than the gooseberry-bushes. How many were there of each?

22. Atmospheric air is composed principally of 2 gases, oxygen and nitrogen, mixed in the proportions of one part of the former to four parts of the latter. How much of each gas is there in a room that is 12 feet broad, 14 feet long, and $7\frac{1}{2}$ feet high?

23. I have three vessels, two small ones and a large one. The first of the two small ones contains but $\frac{3}{16}$ of the large one, and the second only $\frac{5}{24}$; if I fill the second vessel and pour it into the first, which pre-

viously contained nothing, 10 quarts must remain in the second. How many quarts does each of the vessels hold?

24. A farmer had a flock of geese and a flock of sheep, 432 in all. As he wished to give up keeping geese, he exchanged all he had with a neighbour for sheep, obtaining three of the latter for 32 of the former. When this had been done, he found that he had 200 sheep. How many geese had he parted with?

25. A man of pleasure from his eighteenth year to the end of his life had spent $\frac{3}{8}$ of his time in sleep, $\frac{1}{16}$ in eating and drinking, $\frac{1}{4}$ in taking walks, $\frac{3}{16}$ at play, $\frac{1}{16}$ on his sofa, and during the whole worked but 2 years. How old had he become?

26. Pythagoras was one day asked to tell the number of his disciples. "The half," said he, "are devoted to mathematics; a fourth are inquiring into the mysteries of nature; a seventh in silence discipline their minds; and of women there are three, with the noble Theano at their head. Count them for thyself, Polycrates."

27. A boy tending a small flock of sheep was met by a saucy fellow, who said to him, "Give me the half of thy 100 fat sheep." The shepherd replied, "If I had 5 times as many, and $\frac{2}{3}$ and $\frac{3}{4}$ and $\frac{5}{6}$ as many as I have, and thine own woolly head were counted among them, there would be but just 100." How many were there?

CHAPTER VI.

INVOLUTION.

It will be recollected that when a quantity is multiplied into itself, the product is called a power of that quantity; also that powers are denoted by indices or exponents. Involution, or the raising of powers, is only an application of multiplication to cases in which the factors are the same.

To raise the power of a power of a quantity, multiply their indices together. Thus $(a^2)^3=a^6$, *i.e.* $a^2 \times a^2 \times a^2$ or a^{2+2+2}.

The cube of a square gives the same result as the square of a cube, $3\times 2=2\times 3$: and the fourth power of a quantity may be obtained by merely squaring its square; the sixth by squaring its cube; the eighth by squaring its fourth, &c. The third power is found by taking the product of the first and second, that is, of the quantity itself and its square; the fifth by taking that of the square and cube; the seventh by taking that of the third and fourth, and so on.

The second power of $a^n=a^{2n}$, the third power of $(xy)^n=(xy)^{3n}$, and the p^{th} power of $z^n=z^{np}$.

If a binomial $a+b$ is to be raised to the n^{th} power, it will be denoted by $(a+b)^n$. Were that form to be expanded, the result might be very lengthy according to the value of n, as may be seen from the following example;

$$(x+y)^2=x^2+2xy+y^2.$$
$$(x+y)^3=x^3+3x^2y+3xy^2+y^3.$$
$$(x+y)^4=x^4+4x^3y+6x^2y^2+4xy^3+y^4.$$
$$(x+y)^5=x^5+5x^4y+10x^3y^2+10x^2y^3+5xy^4+y^5.$$

It is evident from the above, that the expansion of the second power of a binomial consists of three terms; that of the third power of four terms; that of the fourth of five terms; and that of the fifth of six terms; whence it may be concluded that in general the n^{th} expansion of a power will consist of $n+1$ terms.

It may also be observed that the first and last terms have no coefficients, or rather have the coefficient 1 prefixed; for x^2 is $1x^2$; x^3 is $1x^3$; y^4 is $1y^4$; y^5 is $1y^5$, &c.

It will further be observed that the exponents of the terms in each of the above expansions ascend and descend in a fixed order, and that y presents an arrangement exactly the reverse of x, as the following analysis shows.

$$\begin{array}{c}
x^5 \ldots x^4 \ldots x^3 \ldots x^2 \ldots x^1 \ldots x^0 \\
y^0 \ldots y^1 \ldots y^2 \ldots y^3 \ldots y^4 \ldots y^5 \\
\hline
x^5 \ldots x^4y \ldots x^3y^2 \ldots x^2y^3 \ldots xy^4 \ldots y^5\,;
\end{array}$$

therefore it is easy to write down mechanically the expansion of the terms of any binomial, so far as the indices are concerned.

The coefficients may also be determined by fixed laws, as there is, in the development of their succession, an order easily discovered and reduced to rule.

Rule. To find the coefficient of any term, multiply the coefficient of the preceding term by the exponent of its leading quantity, and divide by the number denoting the place or position of the latter term.

In the example of the expansion of the fifth power referred to above, the coefficient of the second term is 5, because $\frac{5\times1}{1}=5$. The coefficient of the third term is 10, because $\frac{5\times4}{2}=10$. That of the fourth is also 10, because $\frac{10\times3}{3}=10$. That of the fifth term is 5, because $\frac{10\times2}{4}=5$; and that of the sixth term is 1, in the first place because it is a continual product of the second term given, and in the next place because $\frac{5\times1}{5}=1$.

The whole expansion would therefore be written thus:

$$\begin{array}{c}
x^5 \,.\,.\, x^4 \ldots x^3 \ldots\ldots x^2 \ldots\ldots x^1 \,.\,.\, x^0 \\
y^0 \,.\,.\, y^1 \ldots y^2 \ldots\ldots y^3 \ldots\ldots y^4 \,.\,.\, y^5 \\
1 \,.\,.\, 5 \ldots 10 \ldots\ldots 10 \ldots\ldots 5 \,.\,.\, 1 \\
\hline
\text{or, } x^5+5x^4y+10x^3y^2+10x^2y^3+5xy^4+y^5.
\end{array}$$

An expansion of the seventh power of $(x+y)$ would stand thus:

$$\begin{array}{c}
x^7 \,.\,.\, x^6 \ldots x^5 \ldots x^4 \ldots x^3 \ldots x^2 \ldots x^1 \,.\,.\, x^0 \\
y^0 \,.\,.\, y^1 \ldots y^2 \ldots y^3 \ldots y^4 \ldots y^5 \ldots y^6 \,.\,.\, y^7 \\
1 \,.\,.\, 7 \ldots 21 \ldots 35 \ldots 35 \ldots 21 \ldots 7 \,.\,.\, 1 \\
\hline
\end{array}$$

or, $x^7+7x^6y+21x^5y^2+35x^4y^3+35x^3y^4+21x^2y^5+7xy^6+y^7.$

And generally, an expansion of the n^{th} power of $(x+y)$, so far as the exponents are concerned, thus:

$$\begin{array}{c}
x^n \ldots x^{n-1} \ldots x^{n-2} \ldots x^{n-3}, \text{\&c.} \ldots x^{n-n} \text{ or } x^0 \\
y^0 \ldots y^1 \ldots y^2 \ldots y^3, \text{\&c.} \ldots y^n \\
\hline
x^n \ldots x^{n-1}y \ldots x^{n-2}y^2 \ldots x^{n-3}y^3, \text{\&c.} \ldots y^n,
\end{array}$$

and with respect to the coefficients, thus:

1 of the 1st term . . . . x^n,

$\frac{n}{1}$ of the 2d term . . . . $x^{n-1}y$,

$\frac{n.(n-1)}{2}$ of the 3d term . . . . $x^{n-2}y^2$,

$\frac{n}{1}.\frac{n-1}{2}.\frac{n-2}{3}$ of the 4th term . . . . $x^{n-3}y^3$,

$\frac{n}{1}.\frac{n-1}{2}.\frac{n-2}{3}.\frac{n-3}{4}$ of the 5th term . . $x^{n-4}y^4$;

so that $(x+y)^n=x^n+\frac{n}{1}.x^{n-1}y+\frac{n}{1}.\frac{n-1}{2}.x^{n-2}y^2+\frac{n}{1}.\frac{n-1}{2}.\frac{n-2}{3}.x^{n-3}y^3 + \text{\&c.} + y^n.$

If the terms of a binomial are united by the sign $-$, the successive terms of the expansion will be alternately $+$ and $-$. Hence $(x-y)^6$, expanded, becomes

$$\begin{array}{c}
x^6 \,.\,.\, x^5 \ldots x^4 \ldots x^3 \ldots x^2 \ldots x^1 \,.\,.\, x^0 \\
y^0 \,.\,.\, y^1 \ldots y^2 \ldots y^3 \ldots y^4 \ldots y^5 \,.\,.\, y^6 \\
+1 \quad -6 \quad +15 \quad -20 \quad +15 \quad -6 \quad +1 \\
\hline
x^6-6x^5y+15x^4y^2-20x^3y^3+15x^2y^4-6xy^5+y^6.
\end{array}$$

In general the n^{th} power of $(a \pm b)$ is represented thus:

$$a^n \pm \frac{n}{1}.a^{n-1}b+\frac{n}{1}.\frac{n-1}{2}.a^{n-2}b^2 \pm \frac{n}{1}.\frac{n-1}{2}.\frac{n-2}{3}.a^{n-3}b^3 + \text{\&c.} \pm b^n.$$

If the terms of a binomial are accompanied with coefficients, these may immediately be raised to the power required by the terms to which they are affixed.

The 4th power of $2a+3b$ is thus expanded:

$$\begin{array}{c}
16a^4 \,.\,.\, 8a^3 \ldots 4a^2 \ldots 2a^1 \,.\,.\, (2a)^0 \\
(3b)^0 \,.\,.\, 3b^1 \ldots 9b^2 \ldots 27b^3 \,.\,.\, 81b^4 \\
1 \ldots 4 \ldots 6 \ldots 4 \ldots 1 \\
\hline
16a^4+96a^3b+216a^2b^2+216ab^3+81b^4.
\end{array}$$

The foregoing rule will apply also to trinomials and multinomials, if they be first turned into binomials. For example, $a+b+c=a+(b+c)$, and is thus expanded:

$$\{a+(b+c)\}^2=a^2+2a(b+c)+(b+c)^2=a^2+2ab+2ac+b^2+2bc+c^2.$$

$$\{a+(b+c)\}^3=a^3+3a^2(b+c)+3a(b+c)^2+(b+c)^3=a^3+3a^2b+3a^2c+ 3ab^2+6abc+3ac^2+b^3+3b^2c+3bc^2+c^3.$$

EXERCISE XVII.

1. Write down the second power of the product of a, b, and x; the third power of five times the product of p and q.

2. Write down the fourth power of five times the product of 2, b, x, and y.

3. Write down the second power of the quotient of a divided by b; also the second power of the quotient of the sum of a, b, c, and d divided by a.

4. Write down the sum of a and b, added to the product of c and d; this quantity divided by the difference of x and y; the whole raised to the fourth power.

5. Write down ten times the second power of a added to b.

6. Write down in the fourth power the product of a in the second power, and b in the third power.

7. The square of p added to the cube of q is to be written down in the fourth power; also the quotient of s in the third power, divided by r in the second power, is to be raised to the fourth power.

8. Write down the sum of the squares of a and b; also the square of the sum of a and b; also the difference of the squares of x and y divided by the square of their difference.

9. Write down the difference of the third powers of x and y divided by the third power of their difference.

10. How must we write the square of the sum of a and b added to the square root of their sum?

11. How must we write the product of the sum of p and q in the second power, and the square root of their sum?

12. Write down the cube root of the product of a, the second power of b, 15, and the square of b; minus c.

13. Square 16; 25; 36; 72; 95; 99; 101; 125; 176; 189; 199.

14. Square $3ab$; $15x^2y$; $11bcd$; $3a^2b^2$.

15. Square $6x^2y$; $2\frac{1}{2}abc$; $\cdot5ax^2yz^2$; $\frac{1}{3}ab$; $\frac{1}{6}xy^2$; $3\frac{1}{5}a^3b$; $5ax^3y^2$.

16. Raise a^2 to the 2d, 5th, and 8th powers.

17. Raise a to the n^{th}; x^2 to the p^{th}; a^r to the 3d, 6th, and p^{th} powers.

18. Raise x^n to the m^{th}, p^{th}, and s^{th} powers; x^2y^3 to the m^{th} power.

19. Expand $3(ab^2c)^2$; $16(a^3bc)^2$; $(16a^3bc)^2$; $11(x^2y^3z^4)^2$; $(11x^2y^3z^4)^2$.

20. Expand $(6\times3)^2$; $(12\times0{\cdot}5)^2$; $16(4\times3)^2$; $11(2\times5)^2$; $12(3\times2\times4)^2$; $16(3\times5\times4)^2$; and $(16\times3\times5\times4)^2$.

21. Expand $(a \times ab)^2$; $(b \times x^2y)^2$; $10(3x \times 5x)^2$; $12(4xy \times 2xy)^2$; and $\left(\frac{a}{b} \times \frac{b}{c}\right)^2$.

22. Expand $(10+5)^2$; $(20+5)^2$; $(30-5)^2$; $(50+5)^2$; $(60-5)^2$; $(120+5)^2$; $(130+5)^2$; and $(140-5)^2$, in algebraical form.

23. Square $a+b$; $a-b$; $a-1$; $a+1$; $x+y$; $x-y$; $x-1$; $x-3$; $x+2$; $x+3$; $b+c$; and $c+d$.

24. Square $x+2y$; $2x-y$; $3+y$; $x-4$; $x-6$; $2x+6y$; $a-y$; $4x-5a$; $6a+7b$; $xy+yz$.

25. Expand $(a-b)^2$; $3(a+c)^2$; $(a-x)^2$; $(3a-4b)^2$; $(5x+4y)^2$; and $3(xy-6x)^2$.

26. Expand $(a+b+c)^2$; $(b+c+d)^2$; $(x+y+z)^2$; $(a-b-c)^2$; $(b-c-d)^2$; $(x-y+z)^2$.

27. Expand $4(y^3+x)^2$; $-4(x-y)^2$; $16(a+b+c)^2$; $12(-a+b-c)^2$; $4(3x-4y+5z)^2$; $8(\frac{1}{2}a-\frac{1}{4}b-\frac{1}{8}c)^2$.

28. Expand $(x \pm y)^2$; $(y \pm z)^3$; $(a \pm b)^3$.

29. How much is $(a^2b)^2$; $(a^4b^3)^2$; $(a^nb^5)^2$?

30. How much is $(2a^2)^3$; $(3a^3)^3$; $(-3a^9)^3$; $(-4a^4)^3$?

31. How much is $(ax)^3$; $(-a^nb^2)^3$; $(a^4b^3c)^3$; $(b^{pq})^3$?

32. How much is $(-3a^3bc^n)^4$; $(2a^4b^{3n}c)^5$; $(2a^2b^nc^{2p})^4$?

33. Simplify $\left\{\left((a^m)^n\right)^p\right\}^q$.

34. Simplify $\left\{\left((a^{-m})^{-n}\right)^p\right\}^q$.

35. Simplify $(-3p^3q^2s)^8$; $(2\frac{3}{4}a^nb^m)^7$.

36. Simplify $(-a)^{2m}$ and $(-a)^{2m+1}$.

37. Expand $(2a^2-3b)^4$; $(3ac-2bd)^5$; and $(2a^3-3b^2c)^6$.

38. Expand $(a^2-b^2-c^2)^3$; $(7a^2-3ab+4b^2)^3$; and $(2a-\frac{1}{2}b-5c)^3$.

39. Expand $(a+b+c+d)^3$.

40. Raise $1 \pm x$ to the 11th power.

41. Expand and give the 3d term of $(a+b)^5$.

42. Expand and give the 5th term of $(a+b)^{16}$.

43. Expand and give the 6th term of $(a-b)^{30}$.

44. Expand and give the 5th term of $(a^2-b^2)^{12}$.

45. Expand and give the 9th term of $(2ab-cd)^{14}$.

46. Expand and give the middle term of $(a-b)^{16}$.

47. Expand and give the middle term of $(3a^2c-5b^2d)^{12}$.

48. Expand and give the two middle terms of $(a-b)^{17}$.

CHAPTER VII.

EVOLUTION.

EVOLUTION is the reverse of Involution. It is the extraction of roots; and the rules for it are obtained from observation of those employed in the raising of powers.

Supposing the following numbers to be squared,

$$1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9,\ 10,$$

we have

$$1,\ 4,\ 9,\ 16,\ 25,\ 36,\ 49,\ 64,\ 81,\ 100.$$

Let the square roots of these be taken, and we find that

$$\sqrt{1},\ \sqrt{4},\ \sqrt{9},\ \sqrt{16},\ \sqrt{25},\ \sqrt{36},\ \sqrt{49},\ \sqrt{64},\ \sqrt{81},\ \sqrt{100},$$

give $1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9,\ 10;$

and $\because a\times a=a^2$, $\therefore a$ is the square root of a^2;
$\because a\times a\times a=a^3$, $\therefore a$ is the cube root of a^3;
$\because a^2\times a^2=a^4$, $\therefore a^2$ is the square root of a^4;
$\because a^4\times a^4\times a^4=a^{12}$, $\therefore a^4$ is the cube root of a^{12}.

As x^3 is the square root of x^6, because $(x^3)^2=x^6$, so x^2 is the cube root of x^6, because $(x^2)^3=x^6$; $\sqrt{a^4}=a^2$, $\sqrt[4]{16a^8b^4}=2a^2b$, &c.; hence the root of a simple quantity is found by dividing its index by that of the root required.

But if the index of the quantity does not admit of exact division by the index of the root (as $\sqrt[7]{a^3}$, where 3 cannot be divided by 7), then the root cannot be found, and its extraction must be indicated by writing down the quantity with the sign $\sqrt{}$ before it, along with the index of the root sought. Such quantities are denominated irrational or surd quantities.

An even root of a positive quantity is either positive or negative, and has a double sign ($\pm$); an odd root of a quantity has the same sign as the quantity itself; and an even root of a negative quantity cannot be found.

Ex. $\sqrt{\frac{25x^2}{4y^2}}=\pm\frac{5x}{2y}$; $\sqrt[3]{\pm 27a^3b^3}=\pm 3ab$.*

But when we have a surd quantity similar in form to $\sqrt[5]{-32x^5y^5}=-2xy$, we may take the negative sign from under, and place it before the radical.

The quantity would then be written $-\sqrt[5]{32x^5y^5}$; so also $\sqrt[3]{-a^2}=-\sqrt[3]{a^2}$; $\sqrt[5]{x^{10}}-\sqrt[3]{-b^2}=x^2+\sqrt[3]{b^2}$. We cannot do this with the even roots of negative quantities (such as $\sqrt{-a^2}$; $\sqrt[4]{-16}$), which must remain under their present forms. Such expressions are called *imaginary* or *impossible* quantities, and will be treated of hereafter. They differ from surds in this respect, that they have *no real* value, while surds do possess real values, although they are incapable of exact determination.

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* For the proofs of these rules, see pp. 13, 14, 20.

$$\sqrt{9a^4b^2}=\pm 3a^{\frac{4}{2}}b^{\frac{2}{2}}=\pm 3a^2b.$$

$$\sqrt[3]{-8a^6b^9}=-2a^{\frac{6}{3}}b^{\frac{9}{3}}=-2a^2b^3.$$

$$\sqrt[n]{4^nx^ny^{2n}z^{mn}}=\pm 4^{\frac{n}{n}}x^{\frac{n}{n}}y^{\frac{2n}{n}}z^{\frac{mn}{n}}=\pm 4xy^2z^m.$$

If a simple algebraical quantity has a numerical coefficient, its root is extracted by the ordinary operation of arithmetic; but if the quantity is resolvable into two or more simpler quantities, their roots must be found and multiplied into each other: the product will be the root required.

EXERCISE XVIII.

1. Find $\sqrt{a^2}$; $\sqrt{b^4}$; $\sqrt{y^{10}}$; and $\sqrt{c^6}$.
2. Find $\sqrt[3]{-y^6}$; $\sqrt{x^4y^6}$; $\sqrt{a^6b^8}$; $\sqrt{x^6y^{12}}$.
3. Find $\sqrt{(a+b)^4}$; $\sqrt{(a-b)^4}$; $\sqrt{x^6(x^2+y^2)^2}$; $\sqrt{(x+y)^2(x-y)^2}$.
4. Find $\sqrt[3]{a^6b^9}$; $\sqrt[3]{-x^9y^{12}}$; $\sqrt[3]{x^{15}y^{18}}$.
5. Find $\sqrt[3]{b^6}$; $\sqrt{(a+b)^4}$; $\sqrt[3]{a^{3p}b^{6q}}$; $\sqrt[3]{-x^{3r}y^{6p}}$.
6. Find $\sqrt{\frac{a^6}{b^8}}$; $\sqrt{\frac{a^2b^4}{c^4d^6}}$; $\sqrt{\frac{a^{2n}b^{2p}}{c^{4r}d^{6p}}}$; $\sqrt{\frac{(x+y)^4}{(y+x)^{2m}}}$.
7. Find $\sqrt[3]{\frac{y^9}{z^{15}}}$; $\sqrt[3]{-\frac{x^3y^6}{z^9v^{12}}}$; $\sqrt[4]{\frac{a^8b^{20}c^4}{(a+f)^4h^{12}x^{16}}}$.
8. Find $\sqrt[9]{\left(2^{36}a^{45}b^9\times\frac{(a+b)^{27}}{a^9}\right)}$; $\sqrt{\left(\frac{a^2b^2c^2}{d^4}\times\frac{1}{a^6b^8f^4}\right)}$.

It has been said before, that 'The square of a binomial consists of the square of each term, together with twice the product of the two terms.' From this the rule for the extraction of the square root is derived.

In $\sqrt{x^2+16x+64}$, the square root of the first term is x; taking its square from the whole expression, $16x+64$ remains; mentally dividing $16x$ by $2x$, the double of the first term, we find the root of the second term $=8$; multiplying this number 8 into the double of the first term, and into itself, we have $16x+64$; whence the square root of $x^2+16x+64$ is found to be $x+8$. This is usually expressed thus:

$$\begin{array}{lcl}
x & & \sqrt{(x^2+16x+64)}\,(x+8 \\
\underline{\;2\;} & & \underline{x^2\qquad\qquad\quad} \\
2x & & \\
\quad\underline{\;8\;} & & 16x+64 \\
16x+64 & = & \underline{16x+64} \\
 & & \quad\cdot\quad\cdot
\end{array}$$

The square root of $(4a^2+12ab+16ac+9b^2+24bc+16c^2)$ is found in a similar manner.

$$\begin{array}{rcl}
2a & & \sqrt{(4a^2+12ab+16ac+9b^2+24bc+16c^2)}\big(2a+3b+4c \\
\underline{\;\;2\;\;} & & 4a^2 \\
4a & & \overline{+12ab+16ac+9b^2} \\
\underline{\;\;\;3b} & & \\
12ab+9b^2 & = & 12ab\qquad\;\;+9b^2 \\
2a+3b & & \overline{\qquad\qquad\qquad\qquad\qquad} \\
\underline{\;\;\;2} & & \\
4a+6b & & \qquad16ac\qquad+24bc+16c^2 \\
\underline{\qquad\;\;4c} & & \\
16ac+24bc+16c^2 & = & \qquad\underline{16ac\qquad+24bc+16c^2.} \\
 & & \qquad\;\cdot\qquad\;\;\cdot\qquad\;\;\cdot
\end{array}$$

From these examples may be deduced the following rule.

Rule. In the extraction of the square root of any compound quantity, arrange the terms as in multiplication or division, then the first term will be a square of some quantity. Find that quantity, and set it in the quotient as the square root of the first term; next square this root, and place it under the first term of the dividend and subtract. Bring down the two following terms, double the part of the root already found, and write it down to the left as the first portion of a new divisor. By it divide mentally the first term of the new dividend; the result is a second term of the required root, which must be set in the quotient, and also annexed to the new divisor as a second term. Multiply the new divisor by the last found term of the root, and subtract the product from the first remainder. To the new remainder bring down the two following terms of the compound quantity for a new dividend, and proceed until the quantity is exhausted; or, if there be a remainder, the operation has no limit except the degree of accuracy required.

Ex.

$$\begin{array}{rl}
 & a^4+4a^3x+6a^2x^2+4ax^3+x^4\big(a^2+2ax+x^2 \\
 & a^4 \\
2a^2+2ax) & \overline{+4a^3x+6a^2x^2} \\
 & \underline{+4a^3x+4a^2x^2} \\
2a^2+4ax+x^2) & \qquad+2a^2x^2+4ax^3+x^4 \\
 & \qquad\underline{+2a^2x^2+4ax^3+x^4} \\
 & \qquad\;\;\cdot\qquad\;\cdot\qquad\;\cdot
\end{array}$$

For further illustration we give the following examples:

Extract the square root of $9a^2-26a^4+9a^6+16-12a^5+44a^3-40a$. Arranging the terms, we have

$$\begin{array}{rl}
 & \sqrt{(9a^6-12a^5-26a^4+44a^3+9a^2-40a+16)}\big(3a^3-2a^2-5a+4 \\
(3a^3-2a^2)^2= & \underline{9a^6-12a^5+\;\;4a^4}\qquad\{-12a^5\div2(3a^3)\}=-2a^2 \\
 & \qquad\qquad-30a^4\div2(3a^3)=-5a \\
(3a^3-2a^2-5a)^2= & \underline{9a^6-12a^5-26a^4+20a^3+25a^2} \\
 & \qquad\qquad\qquad24a^3\div2(3a^3)=4 \\
(3a^3-2a^2-5a+4)^2= & \underline{9a^6-12a^5-26a^4+44a^3+9a^2-40a+16}\,. \\
 & \qquad\cdot\qquad\cdot\qquad\cdot\qquad\cdot\qquad\cdot
\end{array}$$

After having arranged the terms according to the descending powers of a, the square root was extracted from $9a^6$, which gave $3a^3$ as the first term of the root; next, having divided $-12a^5$ by twice the first term of the root, the second term $-2a^2$ was obtained. Now, the square of the sum of the two terms found was subtracted from the multinomial, and the first term of the remainder $-30a^4$ divided by twice the first term of the root, by which the third term $-5a$ of the root required was found.

To find the fourth term, the square of the sum of the three terms of the root found was subtracted from the multinomial, and the first term of the remainder $24a^3$ again divided by twice the first term of the root, or $6a^3$, by which the fourth term sought, 4, was found.

As now all the terms of the multinomial had been brought into the operation, the work was at an end; and as the square of the sum of the four terms found just equalled the multinomial, it thence appeared that the quantity was a perfect square, and that the union of the terms found, $3a^3-2a^2-5a+4$, was the root required.

In practice, the above operation may be considerably simplified, if we remember that

$$(a+b+c)^2=\{(a+b)+c\}^2=(a+b)^2+2(a+b)c+c^2=(a+b)^2+(2a+2b+c)c,$$
$$(a+b+c+d)^2=\{(a+b+c)+d\}^2=(a+b+c)^2+2(a+b+c)d+d^2=$$
$$(a+b+c)^2+(2a+2b+2c+d)d.$$

The square of a multinomial of n terms consists of the square of the $n-1$ first terms, increased by the product which is obtained when the sum of twice these $n-1$ terms and the n^{th} term is multiplied by this n^{th} term. Hence, if the quantity from which the square root has to be extracted is diminished by the square of a quantity of n terms, it is sufficient if, after having first diminished it by the square of $n-1$ terms, the above product is subtracted from the remainder. The working of the preceding example according to this will be as follows:

$$\begin{array}{rl}
 & \sqrt{}(9a^6-12a^5-26a^4+44a^3+9a^2-40a+16)\big(3a^3-2a^2-5a+4 \\
(3a^3-2a^2)^2= & \underline{9a^6-12a^5+54a^4\qquad\qquad\qquad\qquad} \\
 & \qquad\qquad -30a^4+44a^3+9a^2-40a+16 \\
(6a^3-4a^2-5a)\times -5a= & \qquad\qquad \underline{-30a^4+20a^3+25a^2\qquad\qquad\quad} \\
 & \qquad\qquad\qquad\qquad 24a^3-16a^2-40a+16 \\
(6a^3-4a^2-10a+4)\times 4= & \qquad\qquad\qquad\qquad \underline{24a^3-16a^2-40a+16}
\end{array}$$

The mode of extracting the roots of numbers is the same. The number is divided into periods of two, commencing at the units' place, because from the square of the units, tens come; from that of the tens, hundreds; from the hundreds, tens of thousands; &c.

If the square root is to be extracted from 70225, the number is thus pointed off: 7,02,25. We first take in hundredths the nearest square root from 7 ten-thousandths, which is 2, or 200. The square of 200 or 2 hundredths is 40,000, or 4 ten-thousandths: this is subtracted from 70225, when 30225 remains. We next take the double of 200, and divide the remainder by it mentally, to see how much the second number of the tenths will amount to. This amounts to more than 7 tenths, but because the remainder contains, besides twice the product of the hundredths and the tenths, also the product of the units with both, and their squares, one or two less must generally be taken. Here it amounts to 6, or 6 tenths. The double of 200 multiplied by 60, plus the square of 60, make together 27600, and this subtracted from the remainder 30,225, leaves 2625. If, now, the double of 260,—the sum of two hundredths and six tenths,—is taken, we see directly that the units of the root amount to 5; for dividing 2625 by 520, the nearest multiplier is 5, and if we now multiply the double of 260, or 520, by 5, and to that add the square of 5, we obtain just 2625, from which we see that the square root of 70,225 is 265. This may be thus expressed:

$$\begin{array}{rrl}
200 & \sqrt{7'02'25}\,(200+60+5=265 & \\
\underline{2} & \underline{40000} & \;260 \\
400 & & \;\underline{\;\;2} \\
\underline{60} & 30225 & \;520 \\
24000+3600= & \underline{27600} & \;\underline{\;\;\;5} \\
 & 2625=2600+25 &
\end{array}$$

or, omitting the ciphers,

$$\begin{array}{rrr}
2 & \sqrt{7'02'25}\,(265 & \\
\underline{2} & \underline{4} & \\
46 & 302 & 26 \\
\underline{6} & & \underline{2} \\
276 & \underline{276} & 52\;5 \\
 & 2625 & \underline{5} \\
 & \underline{2625} & 2625 \\
 & \cdot &
\end{array}$$

Again

$$\begin{array}{rr}
 & \sqrt{72'84'11'04'09}\,(85347 \\
8\times 8= & \underline{64} \\
 & 884 \\
165\times 5= & \underline{825} \\
 & 5911 \\
1703\times 3= & \underline{5109} \\
 & 80204 \\
17064\times 4= & \underline{68256} \\
 & 1194809 \\
170687\times 7= & \underline{1194809} \\
 & \cdot\;\cdot\;\cdot
\end{array}$$

In the above examples no remainder is left; but in most cases the square root is found only approximately, and decimals to the hundredth, thousandth, ten-thousandth, &c. part, appended according to the degree of accuracy required. As an instance of this let the square root of 235 be found.

$$\begin{array}{rr}
 & \sqrt{2'35'\cdot000000}(15\cdot329 \\
1\times1= & \underline{1} \\
 & 135 \\
25\times5= & \underline{125} \\
 & 1000 \\
303\times3= & \underline{909} \\
 & 9100 \\
3062\times2= & \underline{6124} \\
 & 297600 \\
30649\times9= & \underline{275841} \\
 & 21759
\end{array}$$

EXERCISE XIX.

Extract the square roots

1. Of 256, 4096, 61009, 582169, 956484, 57198969.
2. Of 68492176, 25836889, 236144689, 1607448649.
3. Of 780811249, 1420913025, 285970396644.
4. Of 41605800625, 48303584206084, 12088868379025.
5. Of 5, 13, 22, 96.

EXERCISE XX.

Extract the square root

1. Of $a^2+2ab+b^2$
2. Of $a^2-2ab+b^2$.
3. Of $a^2-ab+\frac{b^2}{4}$.
4. Of x^2+2x+1.
5. Of $f^6+6f^3x^4+9x^8$.
6. Of $\frac{9a^8}{4}+2a^4n^3+\frac{4n^6}{9}$.
7. Of $\frac{25}{4}a^2b^2-\frac{5}{3}abc^2+\frac{1}{9}c^4$.
8. Of $x^4-ax^3+\frac{1}{4}a^2x^2$.
9. Of $a^{2m}+2a^mx^n+x^{2n}$.
10. Of $a^{2m}-4a^{m+n}+4a^{2n}$.
11. Of $\frac{a^2}{b^2}-\frac{4a}{3c}+\frac{4b^2}{9c^2}$.
12. Of $a^2+2ab+2ac+b^2+2bc+c^2$.
13. Of $9x^2-30ax-3a^2x+25a^2+5a^3+\frac{a^4}{4}$.
14. Of $4x^4+8ax^3+4a^2x^2+16b^2x^2+16ab^2x+16b^4$.

15. Of $9a^2-6ab+30ac+6ab+b^2-10bc-2bd+25c^2+10cd+d^2$.

16. Of $\frac{9}{4}+6x-17x^2-28x^3+49x^4$.

17. Of $9x^4-3ax^3+6bx^3+\frac{a^2x^2}{4}-abx^2+b^2x^2$.

18. Of $\frac{4}{9}a^2x^4-\frac{4}{3}abx^3z+\frac{8}{3}a^2bx^2z^2+b^2x^2z^2-4ab^2xz^3+4a^2b^2z^4$.

19. Of $\frac{a^2-2ab+b^2}{x^4+4ax^2+4a^2}$. 20. Of $\frac{a^2x^2+2ab^2x^3+b^4x^4}{a^{2m}+2a^mx^n+x^{2n}}$.

EXTRACTION OF THE CUBE ROOT.

We have already seen how binomials and multinomials raised to a given power are developed in a certain and fixed order. $(a+b)^3=a^3+3a^2b+3ab^2+b^3$, and this expansion consists of the third power of each term, three times the product of the square of the first term and the second term, and three times the product of the square of the second term and the first term. The same order takes place in every development of power to the third degree, as well in binomials as in multinomials. The last may, by brackets, be reduced to binomials, and thus subjected to the same rule. $(a+b+c)^3=\{a+(b+c)\}^3=a^3+3a^2(b+c)+3a(b+c)^2+(b+c)^3$, which apparently takes another form when it is fully expanded, and the different results collected and arranged according to the order of the alphabet; although in reality it exactly resembles that of a binomial. This invariable form enables us to find out the cubic root of multinomials and numbers.

The method of working cube root is best illustrated by the solution of a few familiar examples. Let us examine the quantity $a^3+3a^2b+3ab^2+b^3$, the cube root of which is known to be $a+b$. The cube root of the first term we find by inspection to be a; if the cube of a be subtracted from the whole quantity, $3a^2b+3ab^2+b^3$ will remain, in which, as the second term of the root is b, must appear, according to what has been already stated, *three times the square of the first term multiplied into the second, plus three times the first term multiplied into the square of the second, plus the cube of the second term.* Now, we find these quantities contained in the above remainder; hence we place b as the second term in the root, subtract, and nothing remains.

We will give the example worked at length, and examine the mode of operation.

$$\begin{array}{lll}
 & & a^3+3a^2b+3ab^2+b^3(a+b \\
3a+b & 3a^2 & a^3 \\ \cline{3-3}
 & (3a+b)b & 3a^2b+3ab^2+b^3 \\ \cline{1-2}
 & 3a^2+3ab+b^2 & 3a^2b+3ab^2+b^3 \\ \cline{3-3}
 & & \cdot \quad \cdot \quad \cdot
\end{array}$$

Take the cube root of a^3, which is a, and tripling it, set on the left of the first remainder; then multiplying it by a, we obtain $3a^2$. Use this as a trial divisor to find b; annex b to $3a$, which gives $3a+b$; multiply this quantity by b, and set the product $(3a+b)b$, or $3ab+b^2$, under the $3a^2$;

add them up, and multiply the sum $3a^2+3ab+b^2$ by b, which gives $3a^2b+3ab^2+b^3$, the quantity required.

Again, to extract the cube root from

$$
\begin{array}{llll}
 & & 8x^6+48x^5y+60x^4y^2-80x^3y^3-90x^2y^4+108xy^5-27y^6 & \\
 & & \underline{8x^6\qquad\qquad\qquad\qquad\qquad\qquad} & (2x^2+4xy-3y^2 \\
6x^2+4xy & 12x^4 & 48x^5y+60x^4y^2-80x^3y^3 & \\
 & \underline{\quad +24x^3y+16x^2y^2} & & \\
 & 12x^4+24x^3y+16x^2y^2 & \underline{48x^5y+96x^4y^2+64x^3y^3\qquad\qquad\qquad} & \\
6x^2+12xy-3y^2 & 12x^4+48x^3y+48x^2y^2 & \quad -36x^4y^2-144x^3y^3-90x^2y^4+108xy^5-27y^6 & \\
 & \underline{\quad -18x^2y^2-36xy^3+9y^4} & & \\
 & \underline{12x^4+48x^3y+30x^2y^2-36xy^3+9y^4} & \quad \underline{-36x^4y^2-144x^3y^3-90x^2y^4+108xy^5-27y^6.} & \\
 & & \qquad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot &
\end{array}
$$

If the student carefully examines this last example, he will find the divisors formed as in the preceding one, and so on when the root contains a greater number of terms.

The mode of proceeding in the extraction of the cube root of any given numerical quantity presupposes a knowledge of the third power of at least the nine numerals.

Number,	1,	2,	3,	4,	5,	6,	7,	8,	9.
Cube,	1,	8,	27,	64,	125,	216,	343,	512,	729.

Further, 1^3, 10^3, 100^3, 1000^3,
equals 1, 1000, 1000000, 1000000000.

Let it be observed from these figures that the cube root of a number having one, two, or three digits is a number of one digit, and that the root of a number with four, five, or six digits is a number of two digits, &c.; hence we dot every third figure of a cube, beginning with the units: the dots show the number of digits in the root.

Suppose we have to find $\sqrt[3]{1953125}$, the operation is as follows:

$$
\begin{array}{lll}
 & & \dot{1}95\dot{3}12\dot{5}\,(125 \\
 & & \underline{1\qquad} \\
32 & 300 & 953 \\
 & \underline{\ \ 64} & \\
 & 364 & \underline{728\qquad} \\
365 & 43200 & 225125 \\
 & \underline{\ 1825} & \\
 & 45025 & \underline{225125} \\
 & & \quad \cdot \quad \cdot
\end{array}
$$

From which it will be clearly seen that the numerical process corresponds exactly with the algebraical, given in the preceding paragraph.

EXERCISE XXI.

Find the cube roots of

1. 12167, 884736, 405224, 2460375, 11089567.
2. 1191016, 17173512, 49836032, 40353607.
3. 64481201, 8000000, 74088000, 318611987.
4. 340068392, 6372783864, 7256313856.
5. 111980168000, 115145914625, 18970074963.
6. 113028882875, 8108486729.

Find the cube roots of

1. $a^3+3a^2b+3ab^2+b^3$.
2. $a^3-3a^2b+3ab^2-b^3$.
3. $x^3+6x^2+12x+8$.
4. $8a^3-84a^2x+294ax^2-343x^3$.
5. $x^6-6cx^5+12c^2x^4-8c^3x^3$.
6. $a^{3m}-6a^{2m+1}x^n+12a^{m+2}x^{2n}-8a^3x^{3n}$.
7. $8-12x^{3n-1}+6x^{6n-2}-x^{9n-3}$.
8. $\frac{a^3c^3}{b^3}x^6-\frac{3a^2c}{b}x^5+\frac{3ab}{c}x^4-\frac{b^3}{c^3}x^3$.
9. $b^3+\frac{3a^2b^2}{2c^2}x^{-2}+\frac{3a^4b}{4c^4}x^{-4}+\frac{a^6}{8c^6}x^{-6}$.
10. $a^3+3a^2b+3a^2c+3ab^2+6abc+3ac^2+b^3+3b^2c+3bc^2+c^3$.
11. $27z^6-54az^5+63a^2z^4-44a^3z^3+21a^4z^2-6a^5z+a^6$.
12. $\frac{a^3y^3}{b^6c^3}+\frac{3a^2cy^4}{b^4d}-\frac{3a^3y^2}{b^4c^2}+\frac{3ac^5y^5}{b^2d^2}-\frac{6a^2c^2y^3}{b^2d}+\frac{3a^3y}{b^2c}+\frac{c^9y^6}{d^3}-\frac{3ac^6y^4}{d^2}+\frac{3a^2c^3y^2}{d}-a^3$.

CHAPTER VIII.

GREATEST COMMON MEASURE.

WHEN one quantity is exactly divisible by another, the latter is said to be a measure of the former. Thus $15a^2c$ is a measure of $45a^3b^2c$, $a-b$ of a^2-b^2, $a+b$ of a^3+b^3, &c.

Such expressions as $45a^3b^2c$, a^2-b^2, and a^3+b^3, which are capable of being thus divided, and which are therefore products of two or more factors, are called commensurable quantities; while those which cannot be measured but by themselves and unity are called incommensurable quantities. The divisors of monomials, such as $3abc$, $15ab^2x$, $13xy(a+b)c^3$, are seen at once.

When two or more monomials or binomials are divisible by the same measure or divisor, this is said to be their *common* measure. When they have more than one common measure, the product of all the common measures forms the *greatest* common measure. If, for instance, we have $6a^2b^3(x-b)(x-c)$, $3a^3b^4(x-b)(x-c)$, and $15a^4b^2(x-b)(x-d)$, the common measures are 3, a^2, b^2, and $x-b$; and the product of these, or $3a^2b^2(x-b)$, is the greatest common measure of the given quantities.

It is often easy to find the greatest common measure of two quantities when each consists of a single term: thus the G. C. M. of a^3b^2c and ab^3c^4 is ab^2c. But when the greatest common measure of multinomials is sought, they must be resolved into their component factors, if such can be found, and the G. C. M. of these taken. For example, for $4x^3y^2+8x^2y^3+4xy^4$ and $4x^4y-4x^2y^3$, we may write

$$4x^3y^2+8x^2y^3+4xy^4=4xy^2(x^2+2xy+y^2)=4xy^2(x+y)(x+y),$$
$$4x^4y-4x^2y^3=4x^2y(x^2-y^2)=4x^2y(x+y)(x-y).$$

These simplified quantities have the terms $4xy$ and $x+y$ common to both, wherefore $4xy(x+y)=4x^2y+4xy^2$ is the greatest common measure.

If, as in the case of $6a^2b+12b^2c$ and $5b^2-5c^2$, which may be resolved into 2, 3, $b(a^2+2bc)$, and $5(b+c)(b-c)$, there are no like factors, a common measure cannot be found.

EXERCISE XXII.

Find the greatest common measure

1. Of $12x^4-12xy^3$ and $4x^3+8x^2y+8xy^2+4y^3$.
2. Of $9x^3y+18x^2y^2+9xy^3$ and $16x^3y-16xy^3$.
3. Of $6a^2b-6b^3$ and $15a^3-30a^2b+15ab^2$.
4. Of $12x^2y-27y^3$ and $8x^3y+24x^2y^2+18xy^3$.
5. Of $80abx^2-125aby^2$ and $144bcx^2-360bcxy+225bcy^2$.
6. Of $5ax+15ay-5bx-15by$ and $10ax+15ay-10bx-15by$.
7. Of x^4-y^4 and x^3-y^3.
8. Of x^5-y^5 and x^4-y^4.
9. Of $6x^5-6y^5$ and $15x^3-15y^3$.
10. Reduce to its lowest terms $\frac{x^2-y^2}{x^3-y^3}$.
11. Reduce to its lowest terms $\frac{x^2-2xy+y^2}{x^2-y^2}$.
12. Reduce to its lowest terms $\frac{5x^3+10x^2y+10xy^2+5y^3}{15x^4-15xy^3}$.

As it is not always easy to resolve a quantity into its factors, and then find their common measure, we are obliged to have recourse to another method, which will now be explained. First, however, it must be observed, that in seeking the greatest common measure of quantities, all the factors which are common to both may be taken out, for the resulting quantities will still have divisors in common, if there are any. The common factors are thrown out only to simplify the operation. For instance, in $a^3+2a^2b+ab^2$ and $a^4+a^3b-a^2b-ab^2$, the factor a may first be separated from both expressions, and then $a^2+2ab+b^2$ and $a^3+a^2b-ab-b^2$ submitted to the process referred to, the common measure found being afterwards multiplied by a. Again, from $6x^3z^2mn-6y^3z^2mn$ and $5x^5y^3+5x^2y^6$, in the first, $6z^2mn$ may be separated, and in the second, $5x^2y^3$, because these have no common measure; then x^3-y^3 and x^3+y^3 are obtained for further examination.

To find the greatest common measure of two polynomials, after they have been simplified as above, and arranged according to the powers of some common letter, divide the one by the other, and the last divisor by the last remainder, until nothing remains; the last divisor will be the G. C. M. Or, if there is a remainder incapable of being resolved into common factors, the quantities must be considered prime to each other, or incommensurable. In commencing the division, that quantity in which the index of the common letter is lowest must be taken for the divisor, and the other for the dividend. Both divisor and dividend should be abridged, wherever possible; sometimes, also, the one or the other may be multiplied by a new quantity, to facilitate division: neither of these changes at all affects the G. C. M.

Ex. Required the G. C. M. of $x^3+x^2y-xy^2-y^3$ and $x^4+x^2y^2-2y^4$.

$$\begin{array}{rl}
x^3+x^2y-xy^2-y^3) & x^4+x^2y^2-2y^4(x \\
 & x^4+x^3y-x^2y^2-xy^3 \\ \hline
\text{dividing by } y, & -x^3y+2x^2y^2+xy^3-2y^4= \\
-x^3+2x^2y+xy^2-2y^3) & x^3+x^2y-xy^2-y^3(-1 \\
 & x^3-2x^2y-xy^2+2y^3 \\ \hline
\text{dividing by } 3y, & 3x^2y-3y^3= \\
x^2-y^2) & -x^3+2x^2y+xy^2-2y^3(-x+2y \\
 & -x^3 \qquad +xy^2 \\ \hline
 & 2x^2y \qquad -2y^3 \\
 & 2x^2y \qquad -2y^3 \\ \hline
 & \cdot \quad \cdot \quad \cdot
\end{array}$$

From the above operation, it appears that x^2-y^2 is the G. C. M.; and on proving the work we find that the quotient from the first expression divided by x^2-y^2 is $x+y$, and that from the second x^2+2y^2, which are again resolvable into common factors.

In proof of the principles upon which the G.C.M. depends, the following are appended:

I. If $A=aM$, $B=bM$, $C=cM$, $D=dM$, in which a, b, c, d are whole numbers, and M, A, B, C, D any like quantities, then

$$A+B+C+D=(a+b+c+d)M.$$

Proof. As M is contained in the quantity A a times, in the quantity

B b times, in the quantity C c times, in the quantity D d times, so M is evidently contained $(a+b+c+d)$ times in the sum $A+B+C+D$; and therefore

$$A+B+C+D+\ldots=(a+b+c+d+\ldots)M;$$

and hence *a quantity which is a measure of several quantities is also a measure of the sum of these quantities.*

II. If $A=aM$, $B=bM$, and $A>B$, when A, B, M are any like quantities, and a, b whole numbers, then

$$A-B=(a-b)M.$$

Proof. As $A>B$, *i. e.* $aM>bM$, evidently $a>b$; and we are justified in adding $a=b+x$, in which x is a whole number. Further, let

$$A-B=D; \text{ then } A=B+D \quad . \; . \; . \; . \; . \; . \quad (1)$$

Because $a=b+x$, therefore $aM=(b+x)M=bM+xM$;

i. e. $\because aM=A,\ bM=B,\ \therefore A=B+xM \quad . \; . \quad (2)$;

consequently by (1) and (2), $B+D=B+xM$;

subtracting B from both sides,

$$B+D-B=B+xM-B;$$

i. e. $D=xM$, or $A-B=xM$. But $a=b+x$;

thus, by subtracting b from both sides, $a-b=b+x-b$, *i. e.* $a-b=x$; consequently

$$A-B=(a-b)M.$$

Hence *a quantity which is a measure of two quantities is also a measure of the difference of these quantities.*

III. If $A=aB$, $B=bM$, in which A, B, M are any like quantities, and a, b whole numbers, then $A=Mab$.

Proof. Because, by the preceding supposition, $B=bM$ and $A=aB$, $\therefore A=bMa$; but $ab=ba$, $\therefore A=Mab$.

Hence, *when a quantity* M *is a measure of a quantity* B, *the quantity* M *is also a measure of every multiple of the quantity* B.

Also, *when the quantity* M *is exactly divisible in* B, *and* B *in* A, *then* M *is always exactly divisible in* A.

IV. A number which exactly divides the remainder and the divisor also exactly divides the dividend.

A number which exactly divides the remainder and the quotient also exactly divides the dividend.

A number which exactly divides the divisor and the dividend also exactly divides the remainder.

A number which exactly divides the quotient and the dividend also exactly divides the remainder.

Proof. If the divisor is denoted by d, the dividend by D, the quotient by q, and the remainder by r, then clearly

$D=dq+r$, and subtracting dq from both sides, $r=D-dq$.

V. In dividing for the G.C.M. of two whole numbers, the last divisor is not only a common measure of the two quantities, but it is their G.C.M.

Proof. Of the two quantities M and N, let N be contained in M a times, and let R be the remainder; also let R be contained in N b times, and Y be the remainder; and let Y be contained in R c times, without any remainder; then

$$\begin{array}{l}
\mathrm{N})\mathrm{M}(a \\
\;\;a\mathrm{N} \\
\hline
\;\;\;\;\mathrm{R})\mathrm{N}(b \\
\;\;\;\;\;\;b\mathrm{R} \\
\hline
\;\;\;\;\;\;\;\;\mathrm{Y})\mathrm{R}(c \\
\;\;\;\;\;\;\;\;\;\;c\mathrm{Y} \\
\hline
\;\;\;\;\;\;\;\;\;\;0.
\end{array}$$

$$\mathrm{M}=a\mathrm{N}+\mathrm{R};\quad \mathrm{N}=b\mathrm{R}+\mathrm{Y};\quad \mathrm{R}=c\mathrm{Y}.$$

As Y is a measure of cY, it is so of R, and also of bR, or of bR+Y; and also of N, and of aN, or aN+R; and also of M. *Therefore* Y *is a measure of* M *and* N; and it is also their G.C.M.; for R=M−aN, Y=N−bR, and a measure of M and N, or of M and aN, must also be a measure of M−aN or R, and also of bR, and of N−bR or Y; *or every measure of* M *and* N *is also a measure of* Y. *Hence the greatest measure is also a measure of* Y, *and* Y *is itself a measure of* M *and* N; *therefore it must be the* G.C.M.

Where the G.C.M. of three quantities is sought, first find that of two of them, and then the G.C.M. of this and the third quantity; or if of four quantities, first find that of three of them, and afterwards the G.C.M. of this and the remaining one. The last measure found is the G.C.M. of all. For instance, let M, N, R be three quantities, Y the G.C.M. of M and N, and Z that of Y and R; then Z is also the G.C.M. of M, N, and R. As any common measure of M and N is a measure also of Y, therefore any measure of M, N, and R must also be a measure of Y and R; also any measure of Y, being equally one of M and N; therefore any measure of Y and R is also a measure of M, N, and R; whence the G.C.M. of Y and R is also the G.C.M. of M, N, and R.

EXERCISE XXIII.

Find the greatest common measure

1. Of $6x^3+6xy^2-6x^2y-6y^3$ and $15x^4+30x^2y^2+15y^4$.
2. Of $x^2+3xy-10y^2$ and $4x^2-7xy-2y^2$.
3. Of $3x^3+16x^2+21x+20$ and $5x^3+23x^2+20x+32$.
4. Of $12a^4b+18a^3b^2-12ab^4-18b^5$ and $10a^3+5a^2b-15b^3$.
5. Of $6a^4+10ab+12a^3+20b$ and $3a^5+5a^2b+5a^4+5ab+12a^3+20b$.
6. Of $qnp^3+3np^2q^2-2npq^5-2nq^4$ and $2mp^2q^2-4mp^4-mp^3q+mpq^3$.

7. Of $6a^2-17ab+12b^2$, $6a^3-17a^2b+22ab^2-15b^3$, and $10a^2-23ab+12b^2$.

8. Of $10ax^5-15ax^4y+19ax^3y^2+13ax^2y^3-21axy^4+10ay^5$ and $20bx^5-10bx^4y-42bx^3y^2+14bx^2y^3+14bxy^4-12by^5$.

9. Of $6x^5-4x^4-11x^3-3x^2-3x-1$ and $4x^4+2x^3-18x^2+3x-5$.

10. Of $2x^4-3x^3-28x^2+75x-91$ and $10x^3-41x^2+43x-77$.

11. Of $36a^2x^6-18a^2x^5-27a^2x^4+9a^2x^3$ and $27a^2x^5-9a^2x^3-18a^2x^4$.

12. Of $2ba^5+2b^3a^3+4b^4a^2$, $2ba^6+6b^4a^3-4b^5a^2$, $2ba^5+2b^2a^4+8b^4a^2$, and $2b^2a^5-2b(b^2-1)a^4+2b^2(2b^2-1)a^3+4b^3a^2$.

LEAST COMMON MULTIPLE.

When one quantity is exactly divisible by another, the former is denominated a multiple of the latter. Thus $5ab$ is a multiple of b, $5a$, $5b$, a, and ab, because all these are contained in $5ab$ without a remainder; $(na+nb)$ of n and $(a+b)$; x^2-y^2 of $x+y$ or $x-y$; &c. Hence such expressions as $5ab$, $na+nb$, x^2-y^2, in regard to their divisors, are called common multiples.

To obtain the common multiple of any numbers, it is only necessary to multiply them together; for naturally this product may be divided without remainder by any of the given numbers of which a multiple was required. It is evident, however, that the *least* common multiple would not be thus obtained. The product of $4a$, $6b$, $9ab^2c$, and $12abc^2$ is $2592a^3b^3c^3$; but all the quantities are divisible into the far simpler expression $36ab^2c^2$. This arises from the circumstance, that among the given expressions there are some which have a common factor that need appear no more than once in the multiple. All are divisible in it; hence much multiplication may be dispensed with.

For single terms we have the following rule.

Rule. First find the L.C.M. of the numerical coefficients, then multiply that by the highest powers of the letter factors which are in each of the given expressions.

If it be required, for instance, to determine the L.C.M. of $4a^2bd$, $6a^3b^2c$, $15abc^2$, and $20a^2bcd$, the L.C.M. of the numerical coefficients is 60; and as in all the expressions a rises no higher than the third power, b and c no higher than the second, and d is without any exponent, or rather has unity as its exponent, the least common multiple will be $60a^3b^2c^2d$. In the same way, we find that the least common multiple of $40ax$, $15a^2b$, $24(a+b)^2$, $20(a-b)x$, and $6a^3b(a+b)x^2y$ is $120a^3b(a+b)^2(a-b)x^2y$.

When the given expressions are polynomials, we must first find out their common measures, if any, resolve them into their original factors, or reduce them to their simplest forms, and then apply the preceding rule.

Suppose x^2+3x+2, x^2-x-6, x^2-1, x^3+1, and x^2+2x+1 be given to find the L.C.M., we perceive that the two first are divisible by $x+2$, which $x+2$ is no divisor of the remaining expressions; that $x+1$ is a divisor of x^2-1 and x^3+1, and also of x^2+2x+1. Now place the factors found in brackets, and instead of the given expressions write

$$\begin{gathered}(x+1)(x+2)\\(x-3)(x+2)\\(x+1)(x-1)\\(x^2-x+1)(x-1)\\(x+1)(x+1) \text{ or } (x+1)^2;\end{gathered}$$

and as all these factors are now indivisible, the general rule may be followed, and we find as the least common multiple required,

$$(x+1)^2(x+2)(x-3)(x-1)(x^2-x+1).$$

EXERCISE XXIV.

Find the least common multiple

1. Of $6abx^2$, $8ab^2cx$, $9abc^2x^2$, $15bx^3y^2$, and $20xy^3z$.
2. Of $4a^2(x-y)$, $6ab^2(x+y)^2$, and $9b^3c(x^2-y^2)$.
3. Of $27ax^2$, $18xy^2$, $12bz^2$, x^2-y^2, and x^3-y^3.
4. Of $x^2+8xy+15y^2$, x^2+3xy, and $3xy+15y^2$.
5. Of $a^2+5ab+6b^2$, a^2+2ab, and $2ab+6b^2$.
6. Of $5a^2b^2$, $4a^3c+8a^2bc-12ab^2c$, $6a^3b+18a^2b^2-6ab^3-18b^4$, and $6a^3d+18ab^2d+6b^3d$.
7. Of $8m^2n^2-8m^4$, $12m^2-12n^2$, $5m^2+10mn+5n^2$, $3m^2+5mn+2n^2$, $m^2-mn+m-n$, and m^2-1.
8. Of $6x^2-5x-6$, $3x^3-x^2-8x-4$, $10x^2-21x+9$, $3x^2-x-2$, and $5x^3+7x^2-x-3$.

CHAPTER IX.

FRACTIONS.

AN algebraic fraction is nothing more than a representation of division. The denominator is the divisor, and the numerator the dividend.

Algebraic fractions may be divided into Proper, Improper, Mixed, and Complex.

Proper fractions are those the index of whose letter of reference, according to which the numerator and denominator are arranged, is higher in the denominator than in the numerator, as $\frac{a-b}{a^2-ab+b^2}$.

When the letter of reference has an index higher in the numerator than in the denominator, the fraction is called improper, as $\frac{a^2+ab+b^2}{a+b}$.

Mixed fractions are those which consist of whole and fractional terms. Of this sort are

$$a+\frac{b^2}{c},\quad a+b-\frac{ab}{a-b},\quad (x+y)^2+(x-y)^2+\frac{x^4+y^4}{x^2-2xy+y^2}.$$

Complex fractions are those in which either the denominator, or the numerator, or both, are fractional or mixed terms. For instance,

$$\frac{2a-3b+\frac{ab}{a-b}}{a^2-b^2};\qquad \frac{p^2-pq+q^2}{p-\frac{pq}{2p+3q}};\qquad \frac{2a+3b-\frac{2ab}{a-b}}{3a^2-4b^2+\frac{2a^3-b^3}{a-2b}}.$$

The following are improper fractions:

$$\frac{A}{1},\ \frac{0}{A},\ \frac{0}{0},\ \text{and}\ \frac{A}{0},$$

wherein A represents any algebraical term.

1. The expression $\frac{A}{1}$ is nothing else than an algebraical whole quantity under the form of a fraction.

2. The improper fraction $\frac{0}{A}$ is evidently *nothing*, for we cannot conceive of any number, positive or negative, whole or fractional, which multiplied into the divisor A would reproduce the dividend 0.

3. The improper fraction $\frac{0}{0}$ represents an indefinite value; for whatever value may be given to the quotient $\frac{0}{0}$, the product of this quotient and the divisor 0 will give 0 again. Let it be observed, however, that whenever an algebraic fraction must be $\frac{0}{0}$, because the numerator and the denominator contain a common factor which $=0$, this fraction will not be always indeterminate, when the common factor is left out of both numerator and denominator. If, for instance, we have $\frac{a^3-1}{a^2-1}$, and assume that $a=1$, the expression equals $\frac{0}{0}$, because the numerator and denominator contain the common factor $a-1=0$; but if this is left out of both, and we write

$$\frac{a^2+a+1}{a+1},\ \text{instead of}\ \frac{a^3-1}{a^2-1}=\frac{(a^2+a+1)(a-1)}{(a+1)(a-1)},$$

then the expression obtained for $a=1$ becomes $1\frac{1}{2}$.

If we have $\frac{x^2-y^2}{x-y}$, or $\frac{xy-xz+yz-y^2}{xy+xz-yz-x^2}$, then by assuming that $x=y=z$, and omitting the factor which is common to the numerator and denominator of this fraction, the first becomes determinate, the last indeterminate. Let the reader try this.

4. The signification of the fractional expression $\frac{A}{0}$ may be manifested thus: Suppose a series of fractions

$$\frac{A}{1},\ \frac{A}{0{\cdot}1},\ \frac{A}{0{\cdot}01},\ \frac{A}{0{\cdot}001},\ \frac{A}{0{\cdot}0001},\ \frac{A}{0{\cdot}00001},\ \&c.$$

of which the values are A, $10A$, $100A$, $1000A$, $10000A$, $100000A$, &c. we see that by leaving the numerator unchanged, and making the denominator ten times smaller, the value of the fraction is increased tenfold. However far we may proceed in this way, and whatever magnitude may be assigned to the fractions, a denominator greater than 0 will always be retained; whence we conclude that whenever the denominator really is 0, the quotient must be of infinite magnitude. This infinite magnitude is represented by the sign ∞, and we thus have $\frac{A}{0} = \infty$, $\frac{1}{0} = \infty$, &c. from which immediately follows $\frac{A}{\infty} = 0$, $\frac{1}{\infty} = 0$, &c.

The improper fractions $\frac{0}{A}$, $\frac{0}{0}$, and $\frac{A}{0}$, are of great importance in estimating the values of algebraic computations: hence it is of importance to understand thoroughly their force and significance.

REDUCTION OF ALGEBRAIC FRACTIONS.

To reduce Fractions to their lowest terms.

Rule. Divide both numerator and denominator by their greatest common measure: the result will be a numerator and denominator prime to each other, and therefore in their lowest terms.

If we take $\frac{91a^2bc}{105abc^2}$, $\frac{4a^2-12ab+9b^2}{4a^2-9b^2}$, and $\frac{x^3+9x^2+26x+24}{x^3+12x^2+44x+48}$, we shall find that the numerators and the denominators are respectively divisible by $7abc$, $2a-3b$, and x^2+6x+8, and therefore

$$\frac{91a^2bc}{105abc^2} = \frac{13a \times 7abc}{15c \times 7abc} = \frac{13a}{15c};$$

$$\frac{4a^2-12ab+9b^2}{4a^2-9b^2} = \frac{(2a-3b)(2a-3b)}{(2a+3b)(2a-3b)} = \frac{2a-3b}{2a+3b}; \text{ and}$$

$$\frac{x^3+9x^2+26x+24}{x^3+12x^2+44x+48} = \frac{(x+3)(x^2+6x+8)}{(x+6)(x^2+6x+8)} = \frac{x+3}{x+6}.$$

EXERCISE XXV.

Reduce to their lowest terms

1. $\frac{396a^3b(x-y)^2}{891ab(x-y)^3x}$.
2. $\frac{3a^5-3a^3}{9a^4+9a^3-18a^2}$.
3. $\frac{95(x^2-y^2)}{114(x^3+y^3)}$.
4. $\frac{11x^2-132x+385}{7x^2+7x-210}$.

5. $\frac{45a^4+18a^3+35a^2+4a-4}{63a^5+40a^3-14a^2-7a+2}$.

7. $\frac{a^2b^3-4a^2c^3-2b^4+8b^2c^2}{a^4b-2a^4c-4a^2b^3+8a^2b^2c+4b^5-8b^4c}$.

6. $\frac{x^2-x-20}{x^2-14x+45}$.

8. $\frac{ax^2-6ax-16a}{ax^2+50ax+96a}$.

To reduce one fraction to another of a given numerator or denominator; for example, to reduce the fraction $\frac{A}{B}$ to another having the denominator C.

Rule. Multiply both numerator and denominator of the given fraction by C, and divide the results by the denominator of the given fraction; then $\frac{A}{B}=\frac{AC}{BC}=\frac{AC+B}{C}$. In the same way,

$$\frac{A}{B}=\frac{AC}{BC}=\frac{C}{BC+A}.$$

If C is a multiple of B and $=BQ$, or $\frac{C}{B}=Q$, then $\frac{A}{B}=\frac{AQ}{BQ}=\frac{AQ}{C}$; that is, to reduce a given fraction to another, the denominator of which is a multiple of the denominator of the one given, divide this multiple by the denominator of the given fraction, then the product of the numerator and this quotient will be the numerator of the fraction required, and the given multiple will be the denominator.

Further, it is evident that to change a given fraction into another, the numerator of which is a multiple of the given fraction, the above process has only to be reversed.

EXERCISE XXVI.

1. Reduce $\frac{a-b}{a+b}$ to a fraction having a^2-b^2 as denominator.
2. Reduce $\frac{c}{d}$ to a fraction having d^2-cd as denominator.
3. Reduce $\frac{x-1}{2x-9}$ to a fraction having $4x^2-81$ as denominator.

To reduce a whole number to a fraction of a given denominator. Divide the whole number by 1 or unity, writing the expression in fractional form; then multiply both numerator and denominator by the number given. If $a+b$ is to be reduced to a fraction of which c is the denominator, we should have

$$a+b=\frac{a+b}{1}=\frac{(a+b)c}{1.c};$$

or, of which b is the denominator, we should have

$$\frac{a+b}{1}=\frac{ab+b^2}{b}, \text{ or } \frac{(a+b)b}{b}.$$

An improper fraction is reduced to a whole or mixed quantity by dividing the numerator by the denominator, and appending the remainder, if any, to the quotient in the form of a fraction. Thus

$$\frac{x^3-3x^2a-3xa^2+a^3}{x-a}=x^2-2xa-5a^2-\frac{4a^3}{x-a}.$$

EXERCISE XXVII.

Reduce to mixed fractions :

1. $\frac{3a^2b-9ab^2-b^3}{2ab}$.
2. $\frac{a^3-2a+3}{a^2+8a-7}$.
3. $\frac{2x^4-6x^3+13x^2-15x+8}{x^2-x+3}$.
4. $\frac{17a^2c^3-3abc^2+c^4-x^4}{-3ac^3}$.

On the other hand, every mixed fraction may be reduced to an improper one. $A\pm\frac{B}{C}$ may be regarded as the result of an operation in division, in which A is the quotient, C the divisor, and B the remainder. As the dividend is obtained when the quotient is multiplied by the divisor, and the remainder added to this product, so the dividend or the numerator will be equal to $AC\pm B$, and we have

$$A\pm\frac{B}{C}=\frac{AC\pm B}{C},$$

from which we derive the following rule.

Rule. To reduce mixed to simple fractions, multiply the whole number by the denominator of the fraction, and add in the numerator. This sum divided by the denominator will be the fraction required. Thus we have

$$3a+\frac{a^2+b^2}{a-b}=\frac{3a(a-b)+(a^2+b^2)}{a-b}=\frac{3a^2-3ab+a^2+b^2}{a-b}=\frac{4a^2-3ab+b^2}{a-b},$$

$$\text{and } 3a-\frac{a^2+b^2}{a-b}=\frac{3a(a-b)-(a^2+b^2)}{a-b}=\frac{3a^2-3ab-a^2-b^2}{a-b}=\frac{2a^2-3ab-b^2}{a-b}.$$

EXERCISE XXVIII.

Reduce the following mixed to simple fractions :

1. $a+\frac{b^2}{a}$.
2. $a-\frac{a(b-c)}{b}$.
3. $a-2b-\frac{2b^2}{a-b}$.
4. $2x-3-\frac{3x-1}{3x^2-x+2}$.
5. $x^2-2x+1-\frac{2x-1}{x^2-3}$.
6. $a^2b^2-ab^3+b^4-\frac{a^3b^2-2b^5}{a+b}$.
7. $17x^2y+\frac{3x^2y^3-7x^4y}{x^2+y^2}$.
8. $5p^2-4pq+2q^2+\frac{8p^3q+11p^2q^2-12pq^3-4q^4}{2p^2-3q^2}$.

Because the quotient multiplied into the divisor produces the dividend, we have, if $\frac{A}{B}=Q$,

$$Q\times B=A, \text{ or } \frac{A}{B}\times B=A, \text{ and } \frac{A}{B}\times nB=nA,$$

applying this to fractions, we observe :

I. Whenever a fraction is multiplied by its denominator, the product is equal to the numerator of the fraction.

II. Whenever a fraction is multiplied by any multiple of its denominator, the product is equal to the same multiple of the numerator.

We have thus the means of reducing all complex fractions to simple ones. If we have

$$\frac{a^2+b^2+\dfrac{a^3-b^3}{a+b}}{a^2-b^2-\dfrac{a^3+b^3}{a-b}}$$

to reduce to a simple fractional expression, we must multiply the numerator and denominator by the least common multiple of the denominators $a+b$ and $a-b$, which is here evidently $(a+b)(a-b)$. Thus this becomes

$$\frac{(a^2+b^2)(a^2-b^2)+(a^3-b^3)(a-b)}{(a^2-b^2)(a^2-b^2)-(a^3+b^3)(a+b)},$$

which expression after development becomes

$$\frac{a^4-b^4+a^4-ab^3-a^3b+b^4}{a^4-2a^2b^2+b^4-a^4-a^3b-ab^3-b^4}=\frac{2a^4-a^3b-ab^3}{-a^3b-2a^2b^2-ab^3}=\frac{a(2a^3-a^2b-b^3)}{-ab(a^2+2ab+b^2)}=$$

$$\frac{2a^3-a^2b-b^3}{-b(a+b)^2}=\frac{(a-b)(2a^2+ab+b^2)}{-b(a+b)^2}=\frac{(b-a)(2a^2+ab+b^2)}{b(a+b)^2}.$$

Note. In the reduction of this fraction it will have been observed that instead of $2a^3-a^2b-b^3$, we may write $a^3-b^3+a^3-a^2b=(a-b)(a^2+ab+b^2)+a^2(a-b)=(a-b)(2a^2+ab+b^2)$; and that because $\frac{p}{-q}=\frac{-p}{q}$, by changing the signs we have written $(b-a)$ in the numerator instead of $-(a-b)$.

EXERCISE XXIX.

Let the following complex fractions be reduced to simple ones:

1. $\dfrac{2a^2-\frac{a}{b}}{3a^2+\frac{a}{b}}$.

2. $\dfrac{a+b+\frac{a^2+b^2}{a+b}}{a-b-\frac{(a+b)^2}{a-b}}$.

3. $\dfrac{3a-\frac{4b^2}{3a}}{9a^2+12ab+4b^2}$.

4. $\dfrac{2x^3-4x^2y-3xy^2+6y^3}{2xy-\frac{3y^3}{x}}$.

5. $\dfrac{2x^2-2x+5}{2x^2-3x-\frac{15}{x-3}}$.

6. $\dfrac{a+\frac{a^2+2b^2}{a+4b}}{1+\frac{2b}{a-b}}$.

7. $\dfrac{7a-\frac{2b^2}{a+b}}{9a-\frac{4b^3}{a^2-b^2}}$.

8. $\dfrac{1+\frac{2xy}{x^2+y^2}}{1-\frac{2xy}{x^2+y^2}}$.

9. $\dfrac{a-3b+\frac{ab^2+5b^3}{a^2+2b^2}}{a-b+\frac{ab^2-b^3}{a^2-2b^2}}$.

Lastly, there remains to show how fractions may be reduced to others of an equal value, having a common denominator.

Rule. Find the least common multiple of the denominators, and multiply each numerator by that factor which is obtained on dividing the least common multiple by its denominator, for a new numerator; the L. C. M. being of course the common denominator.

Let $\frac{p}{4a^2x}$, $\frac{q}{6ax^3y(a+x)}$, $\frac{r}{9y(a^2-x^2)}$, $\frac{s}{15axy(a+x)^2}$ and $\frac{t}{20y(a-x)^2}$ be reduced to a common denominator. We find that $180a^2x^3y(a+x)^2(a-x)^2$ is the L. C. M. of the denominators of the given fractions. Let this be divided by $4a^2x$, the denominator of the first fraction: the quotient is $45x^2y(a+x)^2 \times (a-x)^2$, and multiplying the numerator and the denominator of this fraction by it,

$$\frac{p}{4a^2x}=\frac{45px^2y(a+x)^2(a-x)^2}{180a^2x^3y(a+x)^2(a-x)^2}.$$

If, further, the L. C. M. found is divided by each of the denominators of the following fractions, the quotients will be

$$30a(a+x)(a-x)^2,\ 20a^2x^3(a+x)(a-x),\ 12ax^2(a-x)^2, \text{ and } 9a^2x^3(a+x)^2.$$

and proceeding with these as above, we find

$$\frac{q}{6ax^3y(a+x)}=\frac{30qa(a+x)(a-x)^2}{180a^2x^3y(a+x)^2(a-x)^2},$$

$$\frac{r}{9y(a^2-x^2)}=\frac{20ra^2x^3(a+x)(a-x)}{180a^2x^3y(a+x)^2(a-x)^2},$$

$$\frac{s}{15axy(a+x)^2}=\frac{12sax^2(a-x)^2}{180a^2x^3y(a+x)^2(a-x)^2},$$

$$\text{and } \frac{t}{20y(a-x)^2}=\frac{9ta^2x^3(a+x)^2}{180a^2x^3y(a+x)^2(a-x)^2}.$$

All the given fractions are thus reduced to a common denominator, without the value of any being changed.

EXERCISE XXX.

Reduce to fractions having a common denominator:

1. $\frac{1}{3x^3yz}$, $\frac{1}{6x^2y}$, $\frac{1}{8y^2z}$, and $\frac{1}{15yz^2}$.
2. $\frac{7a^2}{3xy}$, $\frac{7a^2b}{12x^2y}$, $\frac{17b^3}{15xy^3}$, $\frac{8b}{x-y}$, and $\frac{17a}{x+y}$.
3. $\frac{a^2}{a^2-b^2}$, $\frac{b^2}{(a+b)^2}$, and $\frac{ab}{(a-b)^2}$.
4. $\frac{1}{x^2-4}$, $\frac{2}{x^2-4x+4}$, and $\frac{3}{x^2+x-6}$.
5. $\frac{a-b}{x-1}$, $\frac{a+b}{3x-3}$, $\frac{ab}{x^2-1}$, $\frac{a^2}{x^2+2x+1}$, and $\frac{b^2}{3x^2-6x+3}$.
6. $\frac{2a-b}{2a^3-a^2b-b^3}$ and $\frac{a-b}{4a^3+ab^2-b^3}$.

ADDITION AND SUBTRACTION OF ALGEBRAIC FRACTIONS.

In the addition and subtraction of algebraic fractions, which operations are only the reduction of the sums and differences of fractional expressions to a more simple form, we have two cases to consider.

I. When the fractions have the same denominators.

II. When the denominators of the fractions are various.

In the first case it must be recollected that

$$\frac{a+b+c+d}{p}=\frac{a}{p}+\frac{b}{p}+\frac{c}{p}+\frac{d}{p},$$

$$\frac{a-b}{p}=\frac{a}{p}-\frac{b}{p},$$

and $\frac{a-b-c+d-e}{p}=\frac{a}{p}-\frac{b}{p}-\frac{c}{p}+\frac{d}{p}-\frac{e}{p}$; (see p. 20)

and from a consideration of the above we deduce the following rule.

Rule. To find the sum or the difference of fractions with common denominators, find the sum or the difference of their numerators, and divide it by the common denominator; or, add together all the numerators, and under the sum place the common denominator.

For example, if from the sum of the fractions $\frac{5x}{1-x}$ and $\frac{3x-2}{1-x}$, the quantity $\frac{6x-5}{1-x}$ is to be subtracted, instead of

$$\frac{5x}{1-x}+\frac{3x-2}{1-x}-\frac{6x-5}{1-x},$$

following the rule, we write

$$\frac{5x+(3x-2)-(6x-5)}{1-x}=\frac{5x+3x-2-6x+5}{1-x}=\frac{2x+3}{1-x}.$$

In the same way,

$$\frac{a^2+b^2}{p}-\frac{(a-b)^2}{p}+\frac{(a+b)^2}{p}+\frac{a^2}{p}-\frac{b^2}{p}=\frac{a^2+b^2-(a^2-2ab+b^2)+(a^2+2ab+b^2)+a^2-b^2}{p}=$$
$$\frac{a^2+b^2-a^2+2ab-b^2+a^2+2ab+b^2+a^2-b^2}{p}=\frac{2a^2+4ab}{p}=\frac{2a(a+2b)}{p}.$$

Every fraction, the numerator of which is a multinomial, may thus be considered as the sum of two or more fractions; consequently, it may be resolved into several fractions, each of which has a term of the multinomial as numerator, and the common denominator as its denominator. Thus:

$$\frac{3a+2}{a}=\frac{3a}{a}+\frac{2}{a}=3+\frac{2}{a}; \quad \frac{a+b}{ab}=\frac{a}{ab}+\frac{b}{ab}=\frac{1}{b}+\frac{1}{a};$$

$$\frac{x^3-2x^2-3x+4}{y-1}=\frac{x^3}{y-1}-\frac{2x^2}{y-1}-\frac{3x}{y-1}+\frac{4}{y-1}=\frac{x^3-2x^2}{y-1}-\frac{3x-4}{y-1}=$$
$$\frac{x^3}{y-1}-\frac{2x^2+3x-4}{y-1}=\text{\&c.}$$

To make the resolution possible in some cases, and to facilitate it in others, it may be advisable to add a term to the original fraction, or subtract one from it. Had we, for example, $\frac{x^2}{x^2-y^2}$, then it equals

$$\frac{x^2+xy-xy}{x^2-y^2}=\frac{x(x+y)-xy}{x^2-y^2}=\frac{x(x+y)}{x^2-y^2}-\frac{xy}{x^2-y^2}=$$
$$\frac{x(x+y)}{(x-y)(x+y)}-\frac{xy}{x^2-y^2}=\frac{x}{x-y}-\frac{xy}{x^2-y^2}.$$

Had we $\frac{a^2+b^2}{a^2-b^2}$, then by adding to the numerator $2ab$, and taking the same sum from it, we should obtain

$$\frac{a^2+2ab+b^2-2ab}{a^2-b^2}=\frac{(a+b)^2}{a^2-b^2}-\frac{2ab}{a^2-b^2}=\frac{a+b}{a-b}-\frac{2ab}{a^2-b^2}.$$

Note. It would, however, be a great mistake to suppose that when a fraction has a multinomial as denominator, it may be resolved in a similar way. At a glance, the student will see that if for $\frac{a}{b+c}$ he were to write $\frac{a}{b}+\frac{a}{c}$, each of these fractions would be greater than the given one.

When algebraic expressions contain fractions of different denominators, these must first be reduced to a common denominator, and then the rule applied. For example, take the quadrinomial

$$(a-b)+\frac{ab}{2a-b}-\frac{2a^2+b^2}{2a+b}-\frac{b^3}{4a^2-b^2},$$

to reduce to a simple form.

As the fractions are to be brought to a common denominator, first find the least common multiple, which is $4a^2-b^2$. Now, instead of the whole number $a-b$, we may write

$$\frac{(a-b)(4a^2-b^2)}{4a^2-b^2};$$

also,

$$\frac{ab}{2a-b}=\frac{ab(2a+b)}{4a^2-b^2},$$
$$-\frac{2a^2+b^2}{2a+b}=-\frac{(2a^2+b^2)(2a-b)}{4a^2-b^2},$$
$$-\frac{b^3}{4a^2-b^2}=-\frac{b^3}{4a^2-b^2}, \text{ and}$$
$$\frac{(a-b)(4a^2-b^2)}{4a^2-b^2}+\frac{ab(2a+b)}{4a^2-b^2}-\frac{(2a^2+b^2)(2a-b)}{4a^2-b^2}-\frac{b^3}{4a^2-b^2}=$$
$$\frac{(a-b)(4a^2-b^2)+ab(2a+b)-(2a^2+b^2)(2a-b)-b^3}{4a^2-b^2}.$$

And this would be the solution of the example, had we not to inquire whether the fraction can be reduced to a simpler form. We see that the first and third terms of the numerator have $(2a-b)$, in common with the denominator; and for the second and fourth terms, instead of $ab(2a+b)-b^3$, we may write $2a^2b+ab^2-b^3$, or, by adding and subtracting ab^2, $a^2b+2ab^2-ab^2-b^3$, that is, $2ab(a+b)-b^2(a+b)$, or $(2ab-b^2)(a+b)$, or $-b)(a+b)b$.

The fraction found thus becomes

$$\frac{(a-b)(4a^2-b^2)+(2a-b)(a+b)b-(2a^2+b^2)(2a-b)}{4a^2-b^2};$$

and as all the terms of the numerator as well as of the denominator are divisible by $2a-b$, it resolves itself into

$$\frac{(a-b)(2a+b)+b(a+b)-(2a^2+b^2)}{2a+b}=$$

$$\frac{2a^2-ab-b^2+ab+b^2-2a^2-b^2}{2a+b}=\frac{-b^2}{2a+b}=-\frac{b^2}{2a+b}.$$

The preceding operation, performed according to the rules for fractions in common arithmetic, would be as follows;

$$4a^2-b^2=\text{common denominator}$$

$$\begin{array}{lrcr}
\frac{a-b}{1} & (4a^2-b^2)(a-b) & = & +4a^3-4a^2b-ab^2+b^3 \\
+\frac{ab}{2a-b} & (2a+b)ab & = & +2a^2b+ab^2 \\
-\frac{2a^2+b^2}{2a+b} & -(2a-b)(2a^2+b^2) & = & -4a^3+2a^2b-2ab^2+b^3 \\
-\frac{b^3}{4a^2-b^2} & -1\times b^3 & = & -b^3 \\
\hline
 & \text{Sum} & = & \frac{-2ab^2+b^3}{4a^2-b^2}
\end{array}$$

which may be easily reduced to

$$\frac{-b^2(2a-b)}{(2a+b)(2a-b)}=\frac{-b^2}{2a+b}=-\frac{b^2}{2a+b},$$

as we have obtained before.

We add another example of mixed terms, which will, however, require no further explanation.

Required the sum of

$$a+b-\frac{a^2}{b+c},\ a-b+\frac{a^2}{b-c},\ \text{and } 2a-\frac{2b^2c}{b^2-c^2}.$$

$$b^2-c^2=\text{L. C. M.}$$

$$\begin{array}{lrcr}
a+b-\frac{a^2}{b+c} & (b-c)\times -a^2 & = & -a^2b+a^2c \\
a-b+\frac{a^2}{b-c} & (b+c)\times a^2 & = & a^2b+a^2c \\
2a-\frac{2b^2c}{b^2-c^2} & 1\times -2b^2c & = & -2b^2c \\
\hline
\end{array}$$

$$\text{Sum of the fractions }=\frac{2a^2c-2b^2c}{b^2-c^2}=\frac{2c(a^2-b^2)}{b^2-c^2}.$$

As the sum of the whole numbers is $4a$, the whole or required sum

$$=4a+\frac{2c(a^2-b^2)}{b^2-c^2}=\frac{4a(b^2-c^2)+2c(a^2-b^2)}{b^2-c^2}=\frac{4ab^2-4ac^2+2a^2c-2b^2c}{b^2-c^2}.$$

EXERCISE XXXI.

Required the sum of

1. $\frac{x}{a}$ and $\frac{y}{a}$.

2. $\frac{x}{a}$, $-\frac{y}{a}$, $\frac{z}{a}$, $-\frac{v}{a}$, and $-\frac{w}{a}$.

3. $\frac{a^2}{d}$, $\frac{2ab}{d}$, and $\frac{b^2}{d}$.

4. $\frac{7x^2-3}{2x+3}$, $-\frac{2x^2-7}{2x+3}$, and $-\frac{x^2+13}{2x+3}$.

5. $\frac{m^2+n^2}{p-q}$, $-\frac{n^2-3mn}{p-q}$, and $\frac{n^2-5mn}{p-q}$.

6. $\frac{a^2+b^2}{a+2b}+\frac{(a+b)^2}{a+2b}-\frac{(a-b)^2}{a+2b}+\frac{a^2}{a+2b}-\frac{b^2}{a+2b}$.

7. $\frac{3a^4}{a^2+2b^2}+\frac{a^4-a^3b}{a^2+2b^2}+\frac{4a^2b^2-4a^4}{a^2+2b^2}-\frac{3a^3b+ab^3}{a^2+2b^2}$.

8. $\frac{a}{x}+\frac{a}{y}$.

9. $\frac{a}{x}-\frac{a}{y}$.

10. $\frac{a}{b}+\frac{c}{d}$.

11. $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$.

12. $\frac{a^2}{2p}+\frac{b^2}{3q}-\frac{c^2}{4r}+\frac{d^2}{5s}$.

13. $\frac{2xy^2}{3pq}+\frac{3x^2y^2z}{5p^2qr}-\frac{5y^3z^3}{6p^3r^2}-\frac{5}{8}x$.

14. $\frac{a}{a-b}-\frac{b}{a+b}$.

15. $\frac{a+b}{a^2-ab}-\frac{a+b}{b^2-ab}$.

16. $\frac{z}{x+y}-\frac{z-x}{x-y}-\frac{x^2-2yz+y^2}{x^2-y^2}$.

17. $\frac{3}{3x-8}-\frac{4}{4x-9}$.

18. $\frac{5}{x^2}-\frac{3}{x}-\frac{2}{x-1}$.

19. $\frac{3x-4y}{3x-2y}-\frac{2x-6y}{2x-3y}$.

20. $\frac{1}{a}-\frac{1}{b}+\frac{1}{c}-\frac{1}{d}$.

21. $\frac{a}{x-a}+\frac{b}{x-b}+\frac{c}{x-c}$.

22. $\frac{13x-1}{(x-3)^2}+\frac{7}{3-x}+\frac{11}{x^2}$.

23. $\frac{7x^2-3x+6}{(7x-3)^3}-\frac{7x+11}{(7x-3)^2}+\frac{17}{7x-3}$.

24. $\frac{2}{6(x-y)}-\frac{3x}{4(x^2-y^2)}+\frac{4x}{9(x-y)^2}+\frac{5}{12(x+y)}$.

25. $(a-b)+\frac{ab}{a-b}-\frac{a^2+b^2}{a+b}-\frac{a^2b-2ab^2+3b^3}{a^2-b^2}$.

26. $\frac{4}{x-1}+\frac{3}{2x-3}-\frac{2}{4x+3}-\frac{1}{x+1}$.

27. $\frac{3a^2}{5(a-b)}-\frac{3a^2+2b^2}{2(a+b)}-\frac{5a^2}{2(a+b)}-\frac{3ab-b^2}{5(a-b)}+\frac{17a^3+7ab^2}{5(a^2-b^2)}$.

28. $m-n+\frac{n^2}{m+n}$ and $m+n-\frac{mn^2-n^3}{m^2+n^2}$.

29. $2a^2-b^2+\frac{a^3+b^3}{3a-2b}$, $3a^2+b^2-\frac{2(3a^4+2b^4)}{9a^2-4b^2}$, and $5b(b-2a)+\frac{a^3-b^3}{3a+2b}$.

30. $5a+\frac{b^2}{4a}$, $3b-\frac{a^2}{2b}$, $5(a-2b)-\frac{ab+b^2}{9(a-b)}$, and $2a-5b+\frac{a^3-b^3}{2ab}$.

MULTIPLICATION OF FRACTIONS.

The common arithmetical rule, that the product of two or more fractions is equal to the product of their numerators divided by the product of their denominators, is applicable to the case of algebraical fractions.

If, for instance, we have to multiply the fractions

$$\frac{A}{B}, \frac{C}{D}, \frac{E}{F}, \text{ \&c.}$$

in which A, B, C, &c. represent whole terms, we will show that, according to the above rule, their product =

$$\frac{A\times C\times E\times \text{ \&c.}}{B\times D\times F\times \text{ \&c.}}$$

To prove this, suppose that

$$x=\frac{A}{B},\ y=\frac{C}{D},\ z=\frac{E}{F}, \text{ \&c.}$$

then

$$Bx=A,\ Dy=C,\ Fz=E, \text{ \&c.}$$

As, now, the products of equal quantities are equal, the product of the first members of these equations will be equal to the product of the last; so that

$$B\times D\times F\times \text{ \&c.} \times x\times y\times z\times \text{ \&c.} = A\times C\times E\times \text{ \&c.};$$

and dividing every member of this equation by $B\times D\times F\times$ &c. we obtain

$$x\times y\times z\times \text{ \&c.} = \frac{A\times C\times E\times \text{ \&c.}}{B\times D\times F\times \text{ \&c.}}$$

which proves the rule.

The multiplication of algebraic fractional expressions is thus reduced to the multiplication of whole terms; but before proceeding to the application of the given rule, it is of consequence to inquire whether any of the numerators of the given fractions contain a factor which is also to be found in the denominators of the remaining fractions, and which may thus be omitted. If we wait until the multiplication has been performed, the common factors may still be detected in the numerator and denominator of the product; but it is much more expeditious and convenient to omit at the commencement.

If, for instance, $\frac{x^2-1}{x^2-x-6}$ is to be multiplied by $\frac{x^2+x-2}{x^2+2x+1}$, then proceeding according to rule, we find their product to be

$$\frac{(x^2-1)(x^2+x-2)}{(x^2-x-6)(x^2+2x+1)} = \frac{x^4+x^3-3x^2-x+2}{x^4+x^3-7x^2-13x-6};$$

and the G. C. M. being found, we discover the factor which the numerator and denominator of the new fraction have in common; but a little consideration will show that the given product may also be thus written:

$$\frac{(x-1)(x+1)}{(x+2)(x-3)}\times\frac{(x-1)(x+2)}{(x+1)(x+1)}$$

If, now, $x+1$ and $x+2$ are thrown out of the numerators and the denominators, we obtain a much more simplified product:

$$\frac{x-1}{x-3}\times\frac{x-1}{x+1}=\frac{(x-1)^2}{x^2-2x-3}.$$

Again:

$$\frac{7(x^2-y^2)}{8x^2y^2}\times\frac{32xy(3x-2y)}{35(x+y)}\times\frac{2(y-x)}{3x-2y}=-\frac{8(x-y)^2}{5xy};$$

for $7(x^2-y^2)$ being equal to $7(x+y)(x-y)$, and $7(x+y)$ being a factor of the denominator $35(x+y)$ of the second fraction, it may be thrown out, and also $8xy$ and $3x-2y$; whence the given product becomes

$$\frac{x-y}{xy}\times\frac{4}{5}\times\frac{2(y-x)}{1}=\frac{x-y}{xy}\times\frac{4}{5}\times-\frac{2(x-y)}{1}=-\frac{8(x-y)^2}{5xy}.$$

If one of the factors of a multinomial is integral, it must be considered as the numerator of a fraction whose denominator is 1 or unity, and dealt with according to the preceding rules.

$$\frac{a}{b}\times c=\frac{a}{b}\times\frac{c}{1}=\frac{ac}{b};$$

$$x\times\frac{b}{c}\times\frac{d}{e}=\frac{x}{1}\times\frac{b}{c}\times\frac{d}{e}=\frac{xbd}{ce};$$

$$\frac{a^2+ab+b^2}{a^3+b^3}\times(a-b)=\frac{(a^2+ab+b^2)(a-b)}{a^3+b^3}=\frac{a^3-b^3}{a^3+b^3};$$

$$(x^2-y^2)\times\frac{xy}{x+y}\times\frac{x-y}{x^2y^2}=\frac{xy(x^2-y^2)(x-y)}{x^2y^2(x+y)}=\frac{(x-y)^2}{xy}.$$

If one of the factors is a mixed quantity or a complex fraction, it must be reduced to a simple form, and the general rule applied.

Ex. $$\left(a+b+\frac{a^2+b^2}{a-b}\right)\times\left(a-b-\frac{a^2+b^2}{a+b}\right).$$

Since $$a+b+\frac{a^2+b^2}{a-b}=\frac{(a+b)(a-b)+(a^2+b^2)}{a-b}=\frac{2a^2}{a-b},$$

and $$a-b-\frac{a^2+b^2}{a+b}=\frac{(a-b)(a+b)-(a^2+b^2)}{a+b}=-\frac{2b^2}{a+b},$$

the product of the given quantities will be

$$\frac{2a^2}{a-b}\times-\frac{2b^2}{a+b}=-\frac{4a^2b^2}{a^2-b^2}.$$

As we have seen how the product of several fractions may be reduced to the form of a single fraction, we will now show how a fraction may be resolved into its several terms.

1. $\frac{abc}{xyz}=\frac{ab}{xy}\times\frac{c}{z}=\frac{ac}{xz}\times\frac{b}{y}=\frac{a}{x}\times\frac{b}{y}\times\frac{c}{z}=abc\times\frac{1}{xyz}=$, &c.
2. $\frac{pqr}{s}=pq\times\frac{r}{s}=pr\times\frac{q}{s}=p\times\frac{qr}{s}=pqr\times\frac{1}{s}=$, &c.
3. $\frac{a}{bc}=\frac{a}{b}\times\frac{1}{c}=a\times\frac{1}{bc}=$, &c.
4. $\frac{x}{y}=x\times\frac{1}{y}$.

The last form of a resolved fraction is very often employed. We write $\frac{1}{3} \times a$, or $\frac{1}{3}a$, instead of $\frac{a}{3}$; $\frac{1}{5} \times \sqrt{2}$, or $\frac{1}{5}\sqrt{2}$, instead of $\frac{\sqrt{2}}{5}$; $\frac{1}{2}a(3-\sqrt{5})$, instead of $\frac{a(3-\sqrt{5})}{2}$, &c.

EXERCISE XXXII.

Required the products of the following:

1. $\frac{a}{b} \times \frac{c}{d}$.
2. $\frac{3a}{2b^2} \times \frac{8bc}{15y}$.
3. $\frac{27ax^3}{16by^3} \times -\frac{20b^2yz}{9a^2xv}$.
4. $-\frac{2x}{3p} \times \frac{7p^3}{11q^3} \times -\frac{9pq^2}{35xy}$.
5. $\frac{3a^2}{x-3} \times \frac{7b^2}{x-5}$.
6. $\frac{11ab}{x^2-a^2} \times \frac{2(x-a)^2}{33b^3} \times \frac{3b}{2a}$.
7. $\frac{a}{x-a} \times \frac{b}{x-b}$.
8. $\frac{6a^3(c^2-d^2)}{5b(a+b)} \times \frac{7c^2(c-d)}{9a^2b(c+d)^2}$.
9. $\frac{a^2-ab+b^2}{6ab(a^2-b^2)} \times 4a^2(a+b)^2$.
10. $\left\{1+\frac{a}{b}\right\} \times \left\{1+\frac{b}{a}\right\}$.
11. $\left\{3a-\frac{2b^2}{3a}\right\} \times \left\{4a+\frac{7b^2}{4a}\right\}$.
12. $\frac{15m^2n^3(p+q)^2}{(p^2+q^2)^2} \times \frac{2n^3(p^2+q^2)}{5m(p^2-q^2)} \times \frac{m^3}{6n^4pq}$.
13. $\frac{a^2+ab+b^2}{c-d} \times -\frac{c^2-d^2}{a+b} \times -\frac{a^3-b^3}{c+d}$.
14. $\left\{m^2+2mn+\frac{3mn^2+n^3}{m-n}\right\} \left\{2n-m-\frac{5n^2-2m^2}{2n+m}\right\} \left\{m^2-n^2-\frac{4m^4+4n^4}{m^2+n^2}\right\}$.
15. $\frac{2(a^2-b^2)}{5(x+y)} \times \frac{5(a^2+b^2)}{6(x^2-2xy+y^2)} \times \frac{9(x^4-y^4)}{5(a^4-b^4)} \times \frac{3(x-y)^2}{7(x^2+y^2)}$.
16. $\frac{a+b}{x^2y^2(a-b)^2} \times 9x^ny^m(a^2-b^2)$.
17. $\left\{\frac{1}{a}+\frac{1}{b}\right\} \times \left\{\frac{1}{a^2}+\frac{1}{b^2}\right\}$.
18. $\left\{1-\frac{1}{x}\right\} \times \left\{1-\frac{1}{y}\right\} \times \left\{1-\frac{1}{z}\right\}$.

DIVISION OF FRACTIONS.

Here we need only show how the quotient of two simple algebraic fractions is to be derived.

For this purpose, suppose

$$x=\frac{A}{B}, \text{ and } y=\frac{C}{D};$$

then

$$Bx=A, \text{ and } Dy=C.$$

If now both terms of the first equation are multiplied by D, and those of the last by B, we obtain

$$BDx=A \times D, \text{ and } BDy=B \times C;$$

and because equal quantities divided by equal quantities give equal quotients,

$$\frac{BDx}{BDy}=\frac{A\times D}{B\times C}=\frac{A}{B}\times\frac{D}{C}.$$

But

$$\frac{BDx}{BDy}=\frac{x}{y}=x\div y=\frac{A}{B}\div\frac{C}{D};$$

therefore

$$\frac{A}{B}\div\frac{C}{D}=\frac{A}{B}\times\frac{D}{C}.$$

An embodiment of this in words produces the following rule :—Invert the divisor, and proceed as in multiplication.

Suppose the given quotient is put under the form of a complex fraction, then

$$\frac{A}{B}\div\frac{C}{D}=\frac{\frac{A}{B}}{\frac{C}{D}}=\frac{A\times D}{B\times C}=\frac{A}{B}\times\frac{D}{C}.$$

If we have $\frac{3a^2b}{10x^2}$ to divide $\frac{6z}{5xy}$, the operation is

$$\frac{3a^2b}{10x^2}\div\frac{6z}{5xy}=\frac{3a^2b}{10x^2}\times\frac{5xy}{6z}=\frac{a^2by}{4xz}.$$

If the terms are mixed or complex, it is better to reduce them to simple fractions. The two following examples will sufficiently illustrate this :

1. Divide $a+x$ by $x-\frac{a^2}{a-x}$.

$$x-\frac{a^2}{a-x}=\frac{ax-x^2-a^2}{a-x},$$

$$\frac{a+x}{1}\div\frac{ax-x^2-a^2}{a-x}=\frac{a+x}{1}\times\frac{a-x}{ax-x^2-a^2}=\frac{a^2-x^2}{-a^2+ax-x^2},$$

$$\text{or } \because -a^2+ax-x^2=-(a^2-ax+x^2)\,;\ =-\frac{a^2-x^2}{a^2-ax+x^2}.$$

2. Divide $1+\frac{2xy}{x^2+y^2}$ by $1-\frac{2xy}{x^2+y^2}$.

$$1+\frac{2xy}{x^2+y^2}=\frac{x^2+2xy+y^2}{x^2+y^2},$$

$$\text{and } 1-\frac{2xy}{x^2+y^2}=\frac{x^2-2xy+y^2}{x^2+y^2};$$

$$\frac{x^2+2xy+y^2}{x^2+y^2}\div\frac{x^2-2xy+y^2}{x^2+y^2}=\frac{x^2+2xy+y^2}{x^2+y^2}\times\frac{x^2+y^2}{x^2-2xy+y^2}=\frac{x^2+2xy+y^2}{x^2-2xy+y^2}=\frac{(x+y)^2}{(x-y)^2}.$$

As every division, by reversing the form of the divisor, may be reduced to a process of multiplication, so, on the contrary, every fraction may be reduced to the form of division.

Instead of $\frac{ab}{cd}=\frac{a}{c}\times\frac{b}{d}=\frac{b}{d}\times\frac{a}{c}=\frac{a}{d}\times\frac{b}{c}=\frac{1}{cd}\times ab=$ &c.,

we may write, $\frac{a}{c}\div\frac{d}{b}$, or $\frac{b}{d}\div\frac{c}{a}$, or $\frac{a}{d}\div\frac{c}{b}$, or $\frac{1}{cd}\div\frac{1}{ab}$, &c.

Instead of $\frac{a}{b}=\frac{1}{b}\times a=a\times\frac{1}{b}$ we may write $\frac{1}{b}\div\frac{1}{a}$, or $a\div b$, &c.

EXERCISE XXXIII.

Required the quotients of the following :

1. $\frac{a}{b}\div\frac{x}{y}$.
2. $\frac{3ab}{x^2}\div\frac{5xy^2}{xz^2}$.
3. $27a^2b\div\frac{3b^2}{5a^2}$.
4. $\frac{4xy^2}{15x^2}\div 6y^2z$.
5. $\frac{1}{6x^2y^2}\div\frac{1}{9(x+y)}$.
6. $(a^2+ab)\div\frac{a^2-b^2}{x+y}$.
7. $\frac{ab}{a+b}\div\frac{3ab(a-b)}{a^3+b^3}$.
8. $\frac{x+y}{x^2+3xy}\div\frac{x-y}{3x^2+xy}$.
9. $\left\{1+\frac{x}{y}\right\}\div\left\{1+\frac{y}{x}\right\}$.
10. $\frac{3a^2y^2}{3a-2y}\div\frac{5a^3y^2-7a^2y^3}{15a^2+11ay-14y^2}$.
11. $\frac{171(x+y)^2}{19x^2+18xy-y^2}\div 9(x^2-y^2)$.
12. $\frac{3(x^3-y^3)(x+y)}{x^2y^2}\div\frac{7(x^2-y^2)}{5xy}$.
13. $\frac{ap}{bq}\div\left(1+\frac{bp}{aq}\right)$.
14. $\frac{3x^py^q}{10(a+b)^3}\div\frac{6x^3y^2}{25(a+b)^2}$.
15. $\left(p^2+q^2+\frac{3p^2q+pq^2}{p-3q}\right)\div(p^2-q^2)$.
16. $\left(x+\frac{a}{a-b}+\frac{b}{a+b}\right)\div\left(x+\frac{a}{a+b}+\frac{b(3a-b)}{a^2-b^2}\right)$.
17. $\frac{6a^2b+3ax^2}{(p+q)^2-(p-q)^2}\div\frac{6abc+3cx^2}{2p^2q^2}$.
18. $\frac{a^2b^2-c^2+2cd-d^2}{bc-bd}\div\left(\frac{ab}{c-d}-1\right)$.
19. $\left(\frac{x+y}{x-y}-\frac{x-y}{x+y}\right)\div x^2-y^2$.
20. $\left(m+a\times\frac{b+c}{b-c}-\frac{2abc}{b^2-c^2}\right)\div\left(m+a\times\frac{b-c}{b+c}+\frac{2abc}{b^2-c^2}\right)$.

EXERCISE XXXIV.

Miscellaneous Exercises.

1. What is the value of $(3a^2+4b^2)\times(a+b)^2$, if $a=2$ and $b=3$?
2. What is the value of the preceding, when $a=\frac{1}{2}$ and $b=\frac{1}{3}$?

3. What is the value of $(3a^2-2ab+4b^2)\times(a+2b)^2$, if $a=2$ and $b=3$?

4. What is the value of $\{\frac{1}{2}(x+y)-\frac{1}{9}xy\}\times(x^2-y^2)$, when $x=1$ and $y=\frac{3}{4}$?

5. What is the value of $\frac{1}{3}(13-a)+17\times\frac{a-2b}{a+b}-13a^2(\frac{1}{2}a-1)$, when $a=6$ and $b=\frac{1}{2}$?

6. What is the sum of $5x^2-3xy+5y^2$, $7xy-4y^2$, $3x^2+7xy-3y^2$, $4xy-8x^2-7y^2$, and $12x^2-2y^2+3xy$?

7. What is the sum of $2a^4$, $3a^2b^2$, $-4ab^3$, $-5b^4$, $7d^3\sqrt{xy}$, $-6a^2b^2$, $-4a^4$, $24a^4$, $-13a^2b^2$, $2ab^3$, $-3a^3b$, $5d^3\sqrt{xy}$, $7a^3b$, $-2ab^3$, $3d^3\sqrt{ab}$, $7b^4$, $-2e^3\sqrt{ab}$?

8. What is the difference of $7x^3-2x^2y+3xy^2+4y^2-3p^2q+h^2\sqrt{xy}$ and $14x^2y+7y^3-10p^2q+7h^2\sqrt{xy}+4m^3-3x^3$?

9. Subtract $5x^2-2xy-y^2-3a^2b+10abc+2bx-by$ from $xy-ay+3ax+5abc-2a^2b+6x^2-11y^2$.

10. What remains if from the sum of $14a^4-2a^2b^2+5b^4-8x^2y^2$, $12x^2y^2-8a^4+7a^3b-3a^2b^2+2ab^3$, $12a^4-3a^3b+4b^4$, and $3a^4+2a^2b^2$ is subtracted $4a^2b^2-11ab^3+2x^2y^2+5b^4$, $3a^4-2a^2b^2+5a^3b-8b^4+4x^2y^2$, $12ab^3-3a^2b^2$, $10ab^3-2x^2y^2$, and $16a^4-3x^2y^2+5b^4$?

11. Simplify $\frac{1}{2}a-\frac{5}{6}x-(\frac{3}{4}a-\frac{1}{2}x)-(3b+\frac{11}{4}x-\frac{2}{3}a)$.

12. Reduce to the simplest form $3x^2y-2xy^2+x^3-y^3-(3y^3-2x^2y+3xy^2-x^3)+(4x^3+2x^2y)+(4xy^2-3x^3+2y^3-10x^2y)-(5x^2y-4y^3+2x^3)-(7xy^2-3x^3-4y^3+2x^2y)+(12xy^2+6y^3-3x^3-3x^2y)-(8x^3-10xy^2-4x^2y+6y^3)$.

13. How much is $\frac{1}{2}(a+x)^5x^4z^2\times-\frac{4}{5}(a+y)(a+x)^5x^2y^2\times-\frac{15}{8}(a+y)^4$?

14. How much is $(\frac{3}{4}a^5-\frac{3}{7}a^4y+\frac{5}{3}a^3y^2-\frac{2}{3}a^2y^3+\frac{1}{3}ay^4+\frac{5}{6}y^5)\times-\frac{14}{15}a^2y^2$?

15. What is the product of $7x^4-2x^2y^2-3xy^3+4y^4$ and $3x^3-2x^2y+4xy^2+2y^3$?

16. And of x^2+xy+y^2 and x^2-xy+y^2?

17. And of $x^3+x^2y+xy^2+y^3$ and $x^3-x^2y+xy^2-y^3$?

18. Simplify $(\frac{5}{2}x^2+3ax-\frac{7}{3}a^2)\times(2x^2-ax-\frac{1}{2}a^2)$.

19. Divide $12a^5-31a^4b+(20b^2-29c^2)a^3+(37bc^2+18c^3)a^2+(15c^4-24bc^3)a-30c^5$ by $3a^2-4ab-5c^2$.

20. Reduce to products of two factors

$$\begin{aligned}
A&=(ax+by)^2+(ay-bx)^2,\\
B&=(ax+by)^2+(ay-bx)^2+x^2c^2+y^2c^2,\\
C&=(ax+by+cz)^2+(ay-bx)^2+(az-cx)^2+(bz-cy)^2, \text{ and}\\
D&=(ax+by+cz+dv)^2+(ay-bx+cv-dz)^2+(az-bv-cx+dy)^2\\
&\quad+(av+bz-cy-dx)^2.
\end{aligned}$$

21. Reduce to its simplest form the expression
$Y=\frac{2}{3}(x^2+x-1)(x^2-x+1)-\frac{3}{4}(x-2)^2(2x^2-3)+\frac{5}{6}x(x-5)(x+4)(x+1)-\frac{1}{6}(9x^2-13x+10)(4x+5)$.

22. Remove the brackets from $2y\left\{5b(a^2-b^2)-2a\{4x^3-5y^3(x^2-y^2)\}\right\}$.

23. Give the product of $(a^p+b^q-c^r)$ and $(a^p-b^q+c^r)$.

24. Divide $a^2b^4-a^2b^3c-6a^2b^2c^2+5ab^3c^2+2ab^2c^3-abc^4-b^4c^2+c^6$ by $ab^2-3abc+b^2c+c^3$.

25. Divide $2x^{2p}+2x^py^q+4x^pz^r-3x^py-3y^{q+1}-6yz^r$ by $2x^p-3y$.

26. Divide $\frac{2}{5}a^4b-1\frac{61}{75}a^3b^2+1\frac{3}{5}a^2b^3+\frac{3}{10}ab^4-b^5$ by $\frac{3}{5}ab-2b^2$.

27. Divide $\frac{2}{7}a^3-\frac{9}{35}ab^2+\frac{1}{6}a^2b-\frac{3}{20}b^3$ by $\frac{2}{3}a^2-\frac{3}{5}b^2$.

28. Divide (x^5-1) by $(x-1)$.

29. Divide to the 4th term $(1-x^m)\div(1-x)$.

30. Simplify $3a^2b+3ab^2+ac+bc$.

31. Simplify $x^4+x^2y^2+y^4$ and $x^6+x^4y^2-x^2y^4-y^6$.

32. Simplify $\dfrac{(3a^2-0,5a+0,2)^2-(7a^4+a^3-0,55a^2-0,5a)}{2a^2-4a}$.

33. What is the G. C. M. of $-48x^2+120x-72$ and $12x^3-48x^2+60x-24$?

34. What is the G. C. M. of $6a^3-6a^2y+2ay^2-2y^3$ and $12a^2-15ay+3y^2$?

35. Simplify $\dfrac{3}{4(1-x)^2}+\dfrac{3}{8(1-x)}+\dfrac{1}{8(1+x)}-\dfrac{1-x}{4(1+x^2)}$.

36. Simplify $\dfrac{a^2+b^2+c^2+2ab+2ac+2bc}{a^2-b^2-c^2-2bc}$ and $\dfrac{a^2-3ab+ac+2b^2-2bc}{a^2-b^2+2bc-c^2}$.

37. Reduce the following complex fractions to simple ones:

$$\frac{\frac{a}{b}+\frac{c}{d}+\frac{e}{f}}{\frac{g}{h}+\frac{i}{k}+\frac{l}{m}}, \quad \text{and} \quad \frac{\frac{a}{a-b}+\frac{b}{a+b}}{\frac{a}{a-b}-\frac{b}{a+b}}.$$

38. Also $\dfrac{\frac{p}{a}}{\frac{p}{a+v}}$ and $\dfrac{\frac{p}{a}-\frac{p}{a+v}}{\frac{p}{a+v}-\frac{p}{a+2v}}$.

39. What is the sum of $a+b+\dfrac{ab}{a-b}$, $a-b+\dfrac{ab^2}{(a-b)^2}$, and $2a-\dfrac{a^2b}{a^2-2ab+b^2}$?

40. Subtract $\dfrac{a-d}{a^2-b^2}$ from $\dfrac{b+c}{a^2+ab}$.

41. Add $\dfrac{1}{x}-\dfrac{1}{x-y}-\dfrac{1}{x^2+xy+y^2}$.

42. Multiply $a-2b+\dfrac{b^2}{a}$ by $\dfrac{a}{a-b}$, and $x-y+\dfrac{2y^2}{x+y}$ by $x-\dfrac{xy^2-y^3}{x^2+y^2}$.

43. Divide $a^2-b^2+\dfrac{2bcx-c^2}{x^2}$ by $a-b+\dfrac{c}{x}$, and $\dfrac{a^2b^2-x^2+2xy-y^2}{bx-by}$ by $\dfrac{ab}{x-y}-1$.

44. What is the value of $x=6p^2q+2q^3$, when $p=\frac{1}{2}(a+b)$ and $q=\frac{1}{2}(a-b)$?

45. What is the value of $y=\dfrac{c^2-ax}{b}$, if $x=\dfrac{a^2+3c^2}{3a+2c}$?

46. What is the value of $5b^4+10b^3+10b^2+5b+1$, when $b=\dfrac{1}{p-1}$?

47. What is the value of the expressions $\frac{x^4-y^4}{x-y}$, $\frac{x^2-y^2}{(x-y)^2}$, and $\frac{x^2+xy+y^2}{x^3-y^2}$, if $x=y$?

48. What is the G. C. M. of $12a^3+8a^2b+9a^2c+6abc-4ab^2-3b^2c$ and $6a^3+4a^2b-9a^2c-2ab^2-6abc+3b^2c$?

49. Simplify $\dfrac{\left(a+\frac{a^2}{b}\right)\left(b+\frac{b^2}{a}\right)+\left(a-\frac{a^2}{b}\right)\left(b-\frac{b^2}{a}\right)}{\left(a+\frac{b^2}{a}\right)\left(b+\frac{a^2}{b}\right)+\left(a-\frac{b^2}{a}\right)\left(b-\frac{a^2}{b}\right)}$.

50. If in the following expressions,

$$\left\{\frac{a+b}{1-\frac{1}{x}}-\frac{c+d}{x-1}\right\}\times\left\{\frac{a+b}{1+\frac{1}{x}}-\frac{c-d}{x+1}\right\},$$

we substitute $\frac{a+b}{c+d}$ for x, what is their value?

CHAPTER X.

EQUATIONS—(*continued*).

EXERCISE XXXV.

Required the value of x in the following equations:

1. $x+(3a+5b-7c)=4a+3b-4c$.
2. $2=x+0·76544$.
3. $6·789-x=16·665$.
4. $x-(2a-5b+6c)=a+2b-3c$.
5. $p+2s-(2q+4r)=x-(7r-6s)$.
6. $c+3a-x+2b=2a-(b-c)+4b$.
7. $7-\{7+\{7-(7+x)\}\}=7$.
8. $e:x=d$.
9. $x:1·357=0·02468$.
10. $x:-8\frac{2}{3}=-9\frac{3}{4}$.
11. $9\frac{11}{45}:-x=-5\frac{7}{9}$.
12. $a^3+a^2b+ab^2+b^3=\frac{a^4-b^4}{x}$.
13. $a^3-b^3=(a-b)x$.
14. $mx\pm n=p$.
15. $\frac{n}{x}\pm p=q$.
16. $7·77=2·48x-11·4996$.
17. $m^2-mx=n^2-nx$.
18. $1\frac{2}{3}x-99\frac{13}{27}=4\frac{5}{6}-7\frac{8}{9}x$.

19. $\frac{3}{7}x+15-\frac{3}{5}x+29=0.$

20. $mx+n-px-1=nx-x-m+p.$

21. $\frac{1}{2}x-\frac{3}{4}+\frac{5}{6}x-\frac{7}{8}=\frac{9}{10}+\frac{11}{12}x-\frac{13}{14}-\frac{15}{16}x.$

22. $\frac{x}{2}-\frac{x}{3}+\frac{x}{4}-\frac{x}{5}+\frac{x}{6}-\frac{x}{7}+\frac{1}{8}=\frac{x}{9}-\frac{1}{10}.$

23. $x:(a\pm x)=p:q.$

24. $\frac{a}{bx}-c+\frac{d}{ex}-f=\frac{g}{hx}-k+\frac{m}{nx}-o.$

25. $1-\frac{2}{3x}+4-\frac{5}{6x}=7-\frac{8}{9x}+10-\frac{11}{12x}.$

26. $\frac{x}{p+q}-m=n+x.$

27. $a-\frac{m+n}{x}=b-\frac{m-n}{x}.$

28. $2\frac{7}{8}-\{3\frac{7}{8}-(4\frac{1}{4}-4\frac{3}{8}x)\}=6\frac{7}{8}-(7\frac{5}{8}-3\frac{5}{8}x).$

29. $\frac{1}{9}\left\{\frac{1}{7}\left\{\frac{1}{5}\{\frac{1}{3}(x+2)+4\}+6\right\}+8\right\}=1.$

30. $7{\cdot}1-(13{\cdot}4-2{\cdot}5x)4\frac{3}{4}=39{\cdot}7625-(0{\cdot}45+8x)9.$

31. $\frac{5b-6c}{4a^2}x+2a-\frac{5b-4a}{3b-4c}x-\frac{3b-5n}{2a}=\frac{5n-4c}{2a}-\frac{6c-4a}{3b-4c}x.$

32. $2-\frac{5+x}{7}=1-\frac{9-x}{14}.$

33. $a^2b-\frac{a+x}{b}=ab^2-\frac{b+x}{a}.$

34. $\frac{2x-3}{15}-\frac{4x-9}{20}=\frac{8x-27}{30}-\frac{16x-81}{24}-\frac{9}{40}.$

35. $\frac{a^4-b^4}{a^2(a-b)}-\frac{a^2x+b^3}{a^2}=2b+\frac{b^2}{a}.$

36. $3-\left(\frac{4+x}{5}-\frac{6-x}{7}\right)=\frac{8+x}{9}-10.$

37. $\frac{\frac{2}{3}}{\frac{2}{3}+x}-\frac{2}{3}=\frac{2}{3}-\frac{\frac{2}{3}x+\frac{2}{3}}{\frac{2}{3}+x}.$

38. $(x+2):(20-x)=(x+20):(46-x).$

39. $(5x-7):(4x-2)=(15x-125):(12x-97).$

40. $\{(a^2-b^2)x-ab\}\{a-(a+b)x\}=\{(a+b)^2x+ab\}\{b-(a-b)x\}.$

41. $(8-3x)^2+(4-4x)^2=(9-5x)^2.$

42. $\{(a^2-b^2)x-1\}^2+\{2abx-1\}^2=\{(a^2+b^2)x+1\}^2.$

43. $\frac{1+3x}{5+7x}-\frac{9-11x}{5-7x}=14\frac{(2x-3)^2}{25-49x^2}.$

44. $63x-2:\frac{374-77x}{676-143x}=117x-28.$

45. $\frac{1-2x}{3-4x}-\frac{5-6x}{7-8x}=\frac{8}{3}\cdot\frac{1-3x^2}{21-52x+32x^2}.$

46. $\frac{x+2a}{2b-x}+\frac{x-2a}{2b+x}=\frac{4ab}{4b^2-x^2}$.

47. $\frac{x^{n+1}-x^n-x^{n-1}}{2}-2\frac{2x^n+x^{n-1}}{2x-7}=\frac{3x(x^n-x^{n-1})-47x^{n-1}}{6}$.

48. $\frac{4x^{n+1}+3x^{n+2}}{24}+\frac{2x^{n+1}+x^n}{2x-1}=\frac{1}{6}x^{n+1}+\frac{x^{n+2}+24x^n}{8}$.

Further application of Equations of One Unknown Quantity.

Ex. 1. Two gunners throw shells from different mortars. The first had thrown 36 before the second began, he also throws 8 while the latter is throwing 7; but the second uses as much powder for 3 shells as the first does for 4. Required how many shells the second must throw before he has used as much powder as the first.

Let the required number be x; then the first fires within the time $\frac{8}{7}x$ shells, and altogether $(36+\frac{8}{7}x)$ shells. As the number of shells that can be thrown by equal quantities of powder increases in the same proportion that the quantity of powder necessary for every shell decreases, the number of shells is in the inverse ratio of the quantity of powder used for every shot. Because the second for 3 shells requires as much powder as the first for 4 shells, the quantity of powder that the second uses for each shot is in proportion to the quantity used by the first for each shot as 4 to 3.

$$\therefore \quad x : 36+\tfrac{8}{7}x \text{ reversed } =4 : 3,$$
$$\text{or} \quad x : 36+\tfrac{8}{7}x=3 : 4;$$

and as in a proportion the product of the extremes is equal to the product of the means, we have

$$4x=108+\tfrac{24}{7}x,$$
or $x=189$ shells that the second must have fired.

Ex. 2. A man meets three beggars, and to the first gives the half of the coppers in his pocket, and 1 more; to the second he gives half of the remainder, and 1 more; to the third also he gives half of what he has left, and 1 more; after which he finds he has but 3 coppers remaining. How many had he at first?

Let the number he had be	x
then he gives to the first beggar	$\frac{1}{2}x+1$
and he has remaining	$\frac{1}{2}x-1$
to the second he gives $\frac{1}{2}(\frac{1}{2}x-1)+1$, or	$\frac{1}{4}x+\frac{1}{2}$
and he has remaining	$\frac{1}{4}x-\frac{3}{2}$
to the third he gives $\frac{1}{2}(\frac{1}{4}x-\frac{3}{2})+1$, or	$\frac{1}{8}x+\frac{1}{4}$
so that he has remaining at last	$\frac{1}{8}x-\frac{7}{4}$

As he had 3 coppers remaining,

$$\text{then} \quad \tfrac{1}{8}x-\tfrac{7}{4}=3,$$
$$\text{or} \quad x-14=24,$$
$$x=38.$$

EXERCISE XXXVI.

1. A sum of money is put out at $6\frac{1}{4}$ per cent interest. In how many years will the interest amount to $1\frac{5}{8}$ of the capital?

2. At what rate per cent is money lent when the interest for 19 years amounts to $1\frac{3}{10}$ of the capital?

3. There are 1911 acres planted with oaks, beeches, and pines. If of the pines there are 104 acres more than $\frac{7}{15}$ of the space covered by the beeches, and the oaks are spread over 90 acres more than $\frac{7}{9}$ of the beech land, how many acres are there of each?

4. How large was that capital, the interest of which for 8 years amounted to 1914*l.*, when it bore $3\frac{1}{4}$ per cent in the first year, and $\frac{1}{4}$ per cent every following year?

5. A capitalist has invested $\frac{2}{5}$ of his money in railroad shares, $\frac{1}{3}$ in land, and the rest in mining operations. The first yields him 13 per cent, the second 9 per cent, but the third is a loss to him of 3 per cent. Supposing his income to be 888*l.* a-year, what is his capital?

6. A capital of 4800*l.* after a certain number of years amounted to 6972*l.* $\frac{1}{3}$ of the time it was out at $3\frac{1}{2}$ per cent, $\frac{1}{4}$ of the time at $3\frac{3}{4}$ per cent, and the remainder of the time at 4 per cent. How long was it out?

7. A merchant sells two parcels of goods: one for 880*l.*, the other for 720*l.* He allows $1\frac{1}{4}$ per cent more discount on the former than on the latter, and takes 1389*l.* ready money for both. What was the rate of discount on each parcel?

8. The capital employed in a business yielding 50 per cent clear profit doubled itself in 5 years, although 895*l.* were taken out of it every year. What was the sum originally invested?

9. There is a number composed of 6 figures, the last on the left hand being 1. If the 1 be placed first on the right hand, the original number is tripled. What is it?

10. There is a number composed of 6 figures, such that, when the first (which is 2) on the right hand is placed last on the left hand, the new number formed is but $\frac{1}{3}$ of the original one. What is it?

11. Of what number is the 10th part 13 more than the 17th part, after 18 has been taken from the whole?

12. If I multiply a certain number by $7\frac{2}{3}$, subtract the product from $4\frac{5}{6}$, and divide the result by $1\frac{2}{3}$, I obtain $1\frac{7}{8}$. What is the number?

13. A farmer is obliged to sell 60 oxen for want of fodder; his stock is enough for 14 weeks only instead of for 20. How many oxen has he?

14. In the first of two rooms which stand together there are four times as many people as in the second; but if 13 go out of the first into the second, the latter will have in it $1\frac{1}{3}$ times as many as the former. How many are there in the first room?

15. Six little hamlets, A, B, C, D, E, F, lying along one road, and at the following distances apart—namely, A from B, $\frac{3}{8}$; B from C, $\frac{5}{16}$; C from D, $\frac{3}{8}$; D from E, $\frac{1}{4}$; and E from F, $\frac{1}{8}$ of a mile—erect a school at their joint expense, and wish to have it constructed between C and D, so

that the sum of the distances from A, B, and C shall be the same as the sum of the distances from D, E, and F. At what distance from C must the building stand?

16. A father is 30, his son 2 years old. After how many years will the father be 8 times as old as his son, and after how many years 5 times as old? How many years ago was the father 57 times as old as his son?

17. A mother is now 6 times as old as her daughter, and in 5 years will be 3½ times as old. How old is the mother now?

18. Four persons go by the same train from four successive stations, A, B, C, and D, to the same place, E. A is 19 miles from B, B 3 miles from C, and C 5 miles from D. On comparing costs, it appeared that the person from A had paid as much as the three others together. What was the distance from D to E?

19. A woman carried her spun yarn to be woven into linen. The weaver said to her, "If you want 10 ells more than 100, you must bring me 9 hanks more; but if you will take 10 ells less than 100, I can give you 9 hanks back again at once." How many hanks were there?

20. If into an empty cistern 20 gallons of water flow every 3 minutes, after a certain time it will be filled within 40 gallons; but if 52 gallons flow into it every 5 minutes, within the same time 72 gallons will have flowed over. How many gallons of water must be admitted into the cistern per minute in order that in the specified time it may be filled to the brim? and how many gallons does the cistern contain?

21. By working 10 hours a day a mason would finish as many more than 888 cubic feet of work weekly as he now does less than that quantity by working only 8½ hours daily. How many cubic feet does he finish now per week?

22. A merchant gains 8 per cent when he sells his oil at 36*s.* the jar. How much per cent does he gain or lose when he sells the same jar at 32*s.*?

23. When the price of goods is p, there is n per centage profit. How much per cent is gained or lost when the price of the goods is p'?

24. Two friends 78 miles distant from each other agree to meet at a given spot between their places of abode. Both set off at the same time; the one travels 5¼ miles per hour, the other 7¾. When and where will they meet?

25. A messenger going a certain distance every day is despatched on a journey; 5 days after his departure another man is sent, who to catch the first within 8 days is obliged to travel 2½ leagues more per day. How many leagues does the first travel daily?

26. Two messengers start at the same time from A and B to meet somewhere on the road between these places. The one is able to perform the distance in 6¼ hours, the other in 9⅜ hours. At what time will they meet?

27. Two bodies are set in motion at the same time and in the same direction from two points 18 feet asunder. The first moves over 5 feet in 6 minutes, the second over 7 feet in 8 minutes. After how many minutes will the distance between them be 15 feet?

28. How often and at what times will both hands of a watch stand in a direct line and opposite to each other?

29. A hare chased by a hound is 90 leaps in advance, and always takes 5 leaps in the time that the dog takes 4. If 7 leaps of the hare are equal to 5 of the hound, how many leaps must the hound take to catch the hare?

30. A cistern can be filled from 3 taps: from the first in 4 hours, from the second in 10 hours, and from the third in 15 hours. In what time will the cistern be filled if all the taps are kept running together?

31. There are three sluices to a certain reservoir: two for inlet, and one for outlet. When the reservoir is empty, it can be filled in 1¼ days by opening the first sluice, or in 1¾ days by opening the second sluice; but if the reservoir is full, it can be emptied in ¾ day by the third sluice. In how many days would the reservoir be filled if all three sluices were opened together?

32. Two bricklayers have undertaken to do 34 rods of brick-work together; their comparative industry is as 4 : 5, and their strength as 10 : 9. How many rods will each man do?

33. Eight horses in 7 weeks so grazed 400 square rods of land that they not only ate off the fodder preserved on it, but also the grass that grew in the meantime. In the same way, 9 horses grazed 500 square rods in 8 weeks. How many horses could be kept thus for 12 weeks on a meadow of 600 square rods?

34. There are two steam-engines for draining a mine. One raises 11 hhds. every 5 minutes from a depth of 465 feet, the other 31 hhds. every 10 minutes to a height of 264 feet; the two together are of 53 horse-power. What is the horse-power of each?

35. A man has to pay 2007*l.* in 5 months, 3395*l.* in 7 months, and 6740*l.* in 13 months. In how many months should the whole sum of 12,142*l.* be paid equitably?

36. A man has three sums to pay: 1013*l.* at the end of 3½ months, 431*l.* four months later, and the last sum four months after that. What is the last sum, if he pays all three in 6¼ months?

37. A man has 1980*l.* to pay in 5⅓ months. He pays 440*l.* at the end of 3 months, 550*l.* 1½ months later, and 770*l.* 2 months later. How long may he keep the remaining 220*l.* in hand?

38. A man has a certain sum to pay at an appointed time; but he agrees to pay it in 4 equal instalments, between each of which ⅓ of the time is to elapse, and each instalment is to be 100*l.* more than the preceding. What is the sum?

39. For a joint speculation A gives 200*l.* for 4 months, B 160*l.* for 6 months, and C 120*l.* for 8 months; the profits are 136*l.* What is each man's share?

40. Three carriers receive 6*l.* 16*s.* A took 1½ tons 10 miles, B 12 cwt. 15 miles, C 1¼ tons 8 miles. What is each man's share?

41. Four farmers hire a piece of grass-land for 9*l.* A puts a number of cattle on for 12 weeks, B 11 more than A for 10 weeks, and C 50 for 13 weeks. If, now, C pays 4*l.* 17*s.* 6*d.*, how much ought A and B to pay?

42. What numbers must be added to 3 and to 7 to make the ratio of the sums equal to the ratio of 3 and 4?

43. What number must be subtracted from the numerator and denominator of the fraction $\frac{79}{87}$ to make the value of the fraction equal to $\frac{3}{4}$?

44. What number must be added to or subtracted from the terms of the proportion 339 : 355 in order to change the ratio into 21 : 22?

45. What quantity must be added to a and b, or subtracted from them, to make the ratio of their sums or differences equal to $p : q$?

46. A messenger going from A to B finds, after he has been some time on the road, that the distance he has accomplished bears to the rest of the journey the ratio 2 : 3, and that when he has gone 8 miles further the ratio is 6 : 5. How far is A from B?

47. Three gamblers, A, B, and C, play at cards. A brought 10*l.* with him, B 57*l.*, and C 29*l.* After play was over, A's money was to B's as 1 : 3, and C's to the winnings of A was as 3 : 1. How much had C won or lost?

48. In 255 lbs. of a vinous spirit, water and alcohol are mixed in the proportion, by weight, of 2 : 3. How much water must be distilled off to make the proportion of water and alcohol $=3 : 17$?

49. Two numbers, which together make 70, are in a certain ratio to each other. That ratio is reversed when 14 is added to the one number and subtracted from the other. What are the numbers?

50. If I take a certain number of shillings from my purse, and dispose them in the form of a square, 25 more are needed to complete it; but if I take two from each side of the square, I have 31 too many. Required the number.

Equations with Two or more Unknown Quantities.

When there are two unknown quantities in an equation, an infinite number of pairs of values will satisfy it. If $5x+4y=50$, x may have the value of 6, and y the value of 5; but x may also have the value of 2, and y the value of 10; and in both cases the equation $5x+4y=50$ holds good. Yet although two unknown quantities in one equation may have so many solutions, if there is a distinct and independent equation for each unknown quantity, one pair of values only will satisfy both. If, in addition to $5x+4y=50$, we have also $4x+5y=49$, we can by means of the two equations determine the true value of x and y; viz. $x=6$, $y=5$.

The equations $5x+4y=50$, and $4x+5y=49$, which are thus satisfied by the same pair of values of x and y, are called *simultaneous equations.*

Every unknown quantity must have a distinct equation; four unknowns would require four equations, each expressing a relation to the unknown quantity different from the others.

Two equations of the first degree with two unknown quantities are commonly expressed thus:

$$ax+by=c, \text{ and } a'x+b'y=c',$$

in which the coefficients a, b, c, a', b', c' express known quantities; while x and y, the unknown quantities, represent the same value in both. When that value is found and substituted for x and y, it is supposed to make the two equations identical.

There are several ways of solving simultaneous equations containing two unknown quantities, but all resolve themselves into methods for so combining or so multiplying the expressions that one unknown quantity may be severed from the other, or *eliminated*; in which case, of course, an equation with one unknown only remains.

I. Convert the coefficients of the unknown quantity sought to the same value in both equations: this may be done by multiplication or division, as is most convenient. For instance,

1.
$$\begin{array}{l} 2x+5y=48 \\ 3 \\ \hline 6x+15y=144 \end{array} \qquad \begin{array}{l} 3x+4y=44 \\ 2 \\ \hline 6x+8y=88 \end{array}$$

next subtract one from the other,

$$\begin{array}{r} 6x+15y=144 \\ 6x+8y=88 \\ \hline 7y=56, \end{array} \quad \therefore y=\tfrac{56}{7}=8,$$

and substituting 8 for y, the value of x is easily obtained.

2.
$$\begin{array}{l} 5x-3y=90 \\ 2 \\ \hline 10x-6y=180 \end{array} \qquad \begin{array}{l} 2x+5y=160 \\ 5 \\ \hline 10x+25y=800 \end{array}$$

subtracting as before,

$$\begin{array}{r} 10x+25y=800 \\ 10x-6y=180 \\ \hline 31y=620, \end{array} \quad \therefore y=\tfrac{620}{31}=20;$$

then by the first equation $x=\frac{90+3y}{5}=\frac{90+60}{5}=30$.

The value of x might also be found by multiplying the first equation by 5 and the second by 3, and then adding the products, when y would be removed, and we should have $25x+6x=450+480$; whence $x=\frac{930}{31}=30$, as before.

3. $\frac{x}{2}+\frac{y}{3}=16$, or cleared of fractions, $3x+2y=96$. (A.)

$\frac{x}{5}-\frac{y}{9}=2$, „ „ $9x-5y=90$. (B.)

From 3 times A subtract B, and we have

$$11y=198, \text{ or } y=\tfrac{198}{11}=18, \text{ and } x=\frac{96-2y}{3}=20.$$

4. $\frac{x}{2}-12=\frac{y}{4}+8$, or cleared of fractions, $4x-96=2y+64$.

$\frac{x+y}{5}+\frac{x}{3}-8=\frac{2y-x}{4}+27$, or cleared of fractions,

$$12x+12y+20x-480=30y-15x+1620.$$

Simplify the two equations, and we have

$$4x-2y=160, \text{ and}$$
$$47x-18y=2100.$$

Subtract 9 times the former from the latter, and we have

$$11x=2100-1440=660, \quad \therefore x=60,$$
$$\text{and } y=\frac{4x-160}{2}=2x-80=120-80=40.$$

II. Determine the value of one of the unknown quantities in one of the equations, and substitute this value for the same unknown quantity in the other; thence will arise an equation involving only one unknown quantity.

1. Let $x+y=a$, and $bx=cy$;

the first equation gives $x=a-y$. Substituting this value for x in the other, we obtain

$b(a-y)=cy$, or $ab-by=cy$, or $by+cy=ab$; that is, $(b+c)y=ab$, and $y=\frac{ab}{b+c}$.

Now, substituting the value of y in the equation $x=a-y$, we have

$$x=a-\frac{ab}{b+c}=\frac{ab+ac-ab}{b+c}=\frac{ac}{b+c}.$$

2. $$3\times\frac{x+y}{4}-\tfrac{1}{2}(y-\tfrac{1}{2}x)-24=0,$$
$$9(x+y)-40=48-2(x-y).$$

Reducing these equations to their simplest form, we have

$$4x+y=96, \quad \text{and} \quad 11x+7y=88;$$

separating y from the first, we obtain $y=96-4x$; transferring this value of y to the second equation, we obtain

$$11x+7(96-4x)=88, \text{ or } 11x+672-28x=88;$$
$$\text{that is, } 17x=584, \text{ or } x=34\tfrac{6}{17}.$$

If we substitute now this value of x in the equation for the value of y, we have

$$y=96-4\times34\tfrac{6}{17}=-41\tfrac{7}{17}.$$

III. Determine the value of the same unknown in terms of the other in both equations, and put these values equal; proceed for the other unknown quantity as before.

Let $x+y=12$, and $5x+3y=50$;

transposing y and $3y$, we have

$$x=12-y, \text{ and } 5x=50-3y, \text{ or } x=\frac{50-3y}{5}$$

Equating the two values of x, we have

$$12-y=\frac{50-3y}{5}, \quad \therefore 60-5y=50-3y,$$

from which $y=\frac{10}{2}=5$, and $x=12-y=12-5=7$.

Or by II. $x=12-y$. Substituting this value in the second, $60-5y$ must be written instead of $5x$; whence we find

$$60-5y+3y=50, \text{ or } y=\tfrac{10}{2}=5.$$

Or by I. Multiplying the first equation by 5, we obtain

$$\begin{array}{rr} & 5x+5y=60, \\ \left.\begin{array}{r}\text{whence subtracting the} \\ \text{second equation,}\end{array}\right\} & \underline{5x+3y=50,} \\ & 2y=10, \\ \therefore & y=\tfrac{10}{2}=5, \text{ as before.} \end{array}$$

Simultaneous equations of three unknown quantities are solved by reducing them to two equations involving two of the unknown quantities. This is done by eliminating one of the unknowns from any pair of the equations, and the same unknown from any other pair. Having so reduced the three equations to two, proceed for their solution by the methods already given.

EXERCISE XXXVII.

1. $\left.\begin{aligned} x+y&=s,\\ x-y&=d.\end{aligned}\right\}$

2. $\left.\begin{aligned} x+1\tfrac{2}{3}y&=26\tfrac{1}{12},\\ 4\tfrac{5}{6}y-x&=44\tfrac{63}{72}.\end{aligned}\right\}$

3. $\left.\begin{aligned} mx+y&=p,\\ nx+y&=p.\end{aligned}\right\}$

4. $\left.\begin{aligned} 2\tfrac{3}{7}x-\tfrac{3}{4}y&=116,\\ 1\tfrac{3}{5}x-y&=40.\end{aligned}\right\}$

5. $\left.\begin{aligned} 1{\cdot}543689x-y&=1{\cdot}543689,\\ x-0{\cdot}8392867y&=0{\cdot}8392867.\end{aligned}\right\}$

6. $\left.\begin{aligned} \frac{x+5143}{3y+11}&=37,\\ \frac{3262-x}{2y-11}&=43.\end{aligned}\right\}$

7. $\left.\begin{aligned} \frac{x}{a+b}+\frac{y}{a-b}&=\frac{1}{a+b},\\ \frac{x}{a+b}-\frac{y}{a-b}&=\frac{1}{a-b}.\end{aligned}\right\}$

8. $\left.\begin{aligned} \frac{1}{x}&=m-\frac{1}{y},\\ \frac{1}{y}&=\frac{1}{x}-n.\end{aligned}\right\}$

9. $\left.\begin{aligned} \frac{x+a}{n}+y-b&=2a,\\ x+a+\frac{y-b}{a}&=1+na.\end{aligned}\right\}$

10. $\left.\begin{aligned} 1209\tfrac{1}{3}&=60x+77y,\\ 24x-35y&=-152\tfrac{1}{3}.\end{aligned}\right\}$

11. $\left.\begin{aligned} 0{\cdot}077x&=0{\cdot}66y-2{\cdot}1516,\\ 0{\cdot}053y&=0{\cdot}08x+0{\cdot}0842.\end{aligned}\right\}$

12. $\left.\begin{aligned} 1\tfrac{1}{2}x&=1\tfrac{1}{3}y+4\tfrac{5}{12},\\ 4\tfrac{1}{2}x&=\tfrac{1}{3}y-21\tfrac{7}{12}.\end{aligned}\right\}$

13. $\left.\begin{aligned}&(a+b)x-(a-b)y=4ab,\\&(a-b)x+(a+b)y=2a^2-2b^2.\end{aligned}\right\}$

14. $\left.\begin{aligned}&(a+b)x+(c-2b)^2=(b+c)y+a(a-4b),\\&(a-c)x-a(a+b-c)=5bc-4c^2-2b^2-(b-c)y.\end{aligned}\right\}$

15. $\left.\begin{aligned}&10\{x+9\{y-8(x+7)\}\}=6,\\&5\{x+4\{y-3(x+2)\}\}=1.\end{aligned}\right\}$

16. $\left.\begin{aligned}&(x+y):(y-x)=15:8,\\&9x-\frac{3y+44}{7}=100.\end{aligned}\right\}$

17. $\left.\begin{aligned}&(5x+7y):(9x+33)=31:21,\\&(11x+27):(7x+5y)=19:11.\end{aligned}\right\}$

18. $\left.\begin{aligned}&\frac{m}{n+y}=\frac{n}{m-x},\\&\frac{p}{q-x}=\frac{q}{p+y}.\end{aligned}\right\}$

19. $\left.\begin{aligned}&(a+2b)x-(a-2b)y=6ac,\\&(a+3c)y-(a-3c)x=4ab.\end{aligned}\right\}$

20. $\left.\begin{aligned}&(x+1)(y-2)=(3-x)(4-y)-1\\&\frac{2x-3}{4y-5}-\frac{3x-4}{6y-7}=\frac{5}{2(4y-5)(7-6y)}.\end{aligned}\right\}$

21. $\left.\begin{aligned}&2x:y=29:14,\\&y+4x+6=\frac{4y^2+13xy-12x^2}{4y-3x-1}.\end{aligned}\right\}$

22. $\left.\begin{aligned}&\frac{1}{1-x+y}-\frac{1}{x+y-1}=\frac{2}{3},\\&\frac{1}{\frac{1}{1-x+y}\pm\frac{1}{1-x-y}}=-\frac{3}{4}.\end{aligned}\right\}$

23. $\left.\begin{aligned}&3x+5y=161,\\&7x+2z=209,\\&2y+z=89.\end{aligned}\right\}$

24. $\left.\begin{aligned}&53-\tfrac{1}{2}x-\tfrac{1}{2}z=y-109,\\&\tfrac{1}{4}x+\tfrac{1}{8}y=26,\\&5y=4z.\end{aligned}\right\}$

25. $\left.\begin{aligned}&2x-\tfrac{3}{4}y=93-\tfrac{1}{2}x-\tfrac{1}{4}y,\\&7x-5z=y+x-86,\\&\tfrac{1}{2}x+\tfrac{1}{3}y+\tfrac{1}{4}z=58.\end{aligned}\right\}$

26. $\left.\begin{aligned}&2x+5y-7z=-288,\\&5x-y+3z=227,\\&7x+6y+z=297.\end{aligned}\right\}$

27. $\left.\begin{aligned}&\frac{1}{x}+\frac{1}{y}=a,\\&\frac{1}{x}+\frac{1}{z}=b,\\&\frac{1}{y}+\frac{1}{z}=c.\end{aligned}\right\}$

28. $\left.\begin{aligned}&\frac{2}{x}-\frac{5}{3y}+\frac{1}{z}=3\tfrac{4}{27},\\&\frac{1}{4x}+\frac{1}{y}+\frac{2}{z}=6\tfrac{11}{72},\\&\frac{5}{6x}-\frac{1}{y}+\frac{4}{z}=12\tfrac{1}{36}.\end{aligned}\right\}$

29. $\left.\begin{aligned}&\frac{2x+3y}{x+y}=2\tfrac{1}{5},\\&\frac{x+z}{5(x-z)}=\tfrac{1}{3},\\&\frac{10x-3z}{4x-2z}=2\tfrac{9}{14}.\end{aligned}\right\}$

30. $\left.\begin{aligned}&\frac{xy}{ay+bx}=l,\\&\frac{yz}{cz+dy}=m,\\&\frac{xz}{cz+fx}=n.\end{aligned}\right\}$

Exercise XXXVIII.

1. In an assembly of 48 persons, a motion was carried by a majority of 18. How many voted for, and how many against it?

2. One boy said to another, "Give me 5 of your nuts, then I shall have 3 times as many as you." "No," was the reply; "Give me 2 of yours, then I shall have 5 times as many as you." How many had each?

3. A tinker has 2 pots to sell, with but one cover for both; if he sets the cover on the first pot, it is worth as much again as the second; but if he sets the cover on the second pot, it is worth 1⅓ more than the first. If each pot is worth 10*d.* less without the cover, what should each cost?

4. What fraction has the value of ¼ when the numerator is increased by 1, or the value of ⅕ when the denominator is increased by the same?

5. Find a fraction of the value of ½ when numerator and denominator are increased by 1; but of the value of ⅓ when both numerator and denominator are diminished by 1.

6. A and B embark a capital of 10,000*l.* in a business yielding 7½ per cent profit. A leaves his money in it 1 year and 3 months, B leaves his 2 years and 11 months. If the profits are equally divided between them, what should each have invested?

7. The sum of two numbers is 47, their quotient 5, with 5 remainder. What are they?

8. Find two numbers, the sum and quotient of which are a each.

9. If I divide the smaller of two numbers by the larger, the quotient is 0·21, and the remainder 0·04162. If I divide the larger by the smaller, I obtain the quotient 4 and remainder 0·742. What are the two numbers?

10. If I divide one of two numbers by the other, the quotient is $a-b^2$, and the remainder $b+b^4$; if I divide the second number by the first, the quotient is $b-a^2$, and the remainder $a+a^4$. What are the two numbers?

11. A father said to his son, "Seven years ago I was 7 times as old as you, and in three years from this time I shall be 3 times as old as you." How old were both father and son?

12. A certain capital put out at interest at a given rate per cent has, at the expiration of 8 years, increased with the interest to 6486*l.* Had the same capital borne 1 per cent higher interest, in 5 years it would have increased with the interest to 6051*l.* 5*s.* What is the capital, and what the rate of interest?

13. Said A to B, "Sell me ¾ of your interest in this concern, that my share may be worth 100*l.*" "No," replied B; "Let me have ½ of your interest, then my own share will be worth 100*l.*" How much had each?

14. A man had two vessels, and in each a certain quantity of wine. To make them equal he poured out of the first into the second as much as was already in the second; afterwards he poured again out of the second into the first as much as was remaining in the first; and lastly, again out

of the first into the second as much as was still in the second. At the end, there were 80 quarts in each vessel. How many had there been at the beginning?

15. What would be the solution of the preceding question, were the process to be once more repeated, and at last n quarts remained?

16. The sum of 3 times one number and 7 times another is equal to 58; 7 times the first number and 3 times the second is equal to 42. What are the two numbers?

17. Find two numbers, such that the first is to the second as their sum : 5, and as their difference : 3.

18. What would be the solution of the preceding question, if for 5 and 3, the general signs a and b were substituted?

19. A man has two vessels full, and one empty and larger than each of the other two. To fill the empty one, either the contents of the first, together with a fifth of the contents of the second, are necessary, or the contents of the second, together with a third of the contents of the first. The three together would contain 1440 gallons. What would each hold?

20. Two bodies are distant d feet. If they are moved towards each other with equal swiftness they will meet in m seconds; if they are moved, however, with the same swiftness one after the other, they will meet after n seconds. How many feet must each body move over in a second?

21. Two bodies, B and C, approach each other from two points, distant d feet. If B sets off t seconds sooner than C, they meet in m seconds after the departure of C; if, however, C sets off u seconds sooner than B, they meet in n seconds after the departure of B. How many feet does each of the bodies move over in a second?

22. Two merchants, A and B, have been engaged in speculations together at three different times. In the first affair A lent his capital for 4 months, and B his for 5; the profits were 3458*l.* In the second affair A lent his capital for 7 months, and B his for 4 months; the gain was 3591*l.* In the third affair A lent his capital for 9 months, and B his for 11 months; the joint profit was 7651*l.* Supposing that both men have profited to the same amount in the three affairs, what capital did each employ?

23. To a certain metallic compound weighing 300 lbs., and consisting of 2 parts zinc, 3 parts copper, and 4 parts tin, 200 lbs. of another compound consisting of the same materials were added. In an analysis of the mixture of the two compounds are found 3 parts zinc, 4 parts copper, and 5 parts tin. What were the proportions of the 200 lbs. added?

24. A farmer has a certain number of oxen, and fodder for a given number of days. If he sells 75 oxen, his fodder will last 20 days longer, but if he adds 100 more to his stock, the fodder will last 15 days less time. How many oxen has he, and fodder for how many days?

25. A man has three garments made of the same size. In the second he uses 3 yards more than in the first, because the cloth is 2 qrs. narrower; the third, on the contrary, requires 4 yards less than the second, because the cloth is 3 qrs. broader. How much cloth is wanted for the first garment, and of what breadth?

26. Find two numbers whose difference is a times, and product b times as great as their sum.

27. Find two numbers whose sum, difference, and product are in the ratio of $5:1:18$.

28. Four gamblers play four games at cards with each other. In the first game A, B, and C win, and each as much as he possesses; in the second game A, B, and D win, and again, each as much as he possesses; in the third game A, C, and D the same; and in the fourth game B, C, and D. When they count their money they find that each has *5s. 4d.* How much had each before he began?

29. In each of 7 baskets there is a certain number of apples. If out of the first basket I put into all of the others as many as each previously contained; again, out of the second into all of the others as many as each previously contained, and so on, unto the last, there will be the same number in each, viz. 128 apples. How many were in each basket before the division?

30. Find n numbers of such properties, that if the first parts with as much to all the others as each of them consists of in itself, and the second gives as much to all the others as each of them then consists of, and so on to the n^{th}, at last n numbers shall arise all equal to a.

GENERAL THEORY OF POWERS.

A. *Powers with Positive Whole Exponents.*

1. A power with a positive whole exponent is a product of as many like factors as the exponent contains units. Thus a^n designates the n^{th} power of a; *i. e.* a product of n like factors, each of which is a.

2. The root of the n^{th} degree of a quantity A is a quantity which, when raised to the n^{th} power, produces the quantity A. If $a^n=A$, then a is called the n^{th} root of A, and would be denoted by $a=\sqrt[n]{A}$; n is called the exponent of the root.

3. *To multiply powers of the same quantity we must add their indices,*

$$a^m \,.\, a^n=a^{m+n}.$$

For a^m is a product of m factors, each of which is a; and a^n is a product of n factors, each of which is a; $\therefore a^m \,.\, a^n$ is a product of $m+n$ factors, each of which is a. Such a product is a^{m+n}.

4. By (3) we obtain $a^m \,.\, a^n \,.\, a^p \,.\, a^q=a^{m+n+p+q}$.

5. If $m>n$, then $a^m \div a^n=a^{m-n}$.

For, according to (2), $a^{m-n} \,.\, a^n=a^{m-n+n}=a^m$; and $\therefore$ by the principle of division, $a^m \div a^n=a^{m-n}$.

6. $(abcde\ldots.)^n=a^nb^nc^nd^ne^n\ldots.$

For, by (1), $(abcd\ldots)^n=(abcd\ldots)(abcd\ldots)(abcd\ldots)\ldots.,$ in which the number of factors $(abcd\ldots)=n$; but as, however, in every product the order of the factors may be changed at will,

$$\therefore (abcd\ldots)^n=aaa\ldots bbb\ldots ccc\ldots ddd\ldots=a^nb^nc^nd^n\ldots \quad (1)$$

7. $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$.

For, by (1), $\left(\frac{a}{b}\right)^n = \frac{a}{b} \cdot \frac{a}{b} \cdot \frac{a}{b} \cdot \frac{a}{b} \cdots = \frac{aaaa \ldots}{bbbb \ldots}$, in which the number of factors in both numerator and denominator $= n$; $\therefore \left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$.

8. $(a^m)^n = a^{mn}$.

For, by (1), $(a^m)^n = a^m . a^m . a^m . a^m \ldots$, in which the number of factors a^m is n;

$$\therefore (4),\ (a^m)^n = a^{m+m+m+m \cdots} = a^{mn}.$$

9. And hence $(((a^m)^n)^p)^q = a^{mnpq}$.

10. If m is divisible by n, then

$$\sqrt[n]{a^m} = a^{\frac{m}{n}}.$$

For since m is divisible by n, then $\frac{m}{n}$ is a whole number; and because now $m = n\left(\frac{m}{n}\right)$, and by (8), $\left(a^{\frac{m}{n}}\right)^n = a^{n\left(\frac{m}{n}\right)} = a^m$, $\therefore$ (2), $a^{\frac{m}{n}} = \sqrt[n]{a^m}$.

11. These six formulæ embrace the whole theory of indices, and are therefore to be carefully noted:

I. $a^m . a^n = a^{m+n}$.

II. $a^m \div a^n = a^{m-n}$.

III. $(abcd \ldots)^n = a^n b^n c^n d^n \ldots$.

IV. $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$.

V. $(a^m)^n = a^{mn}$.

VI. $\sqrt[n]{a^m} = a^{\frac{m}{n}}$.

B. *Powers with Negative Integral Exponents.*

12. We have seen that $a^m \div a^n = a^{m-n}$, m being $=$ or $>$ n. Now, if we consider a case in which $m < n$ (as in $a^3 \div a^7$), we shall obtain $a^{3-7} = a^{-4}$, an expression in which the exponent is negative.

Since now $a^3 \div a^7 = \frac{a^3}{a^3 . a^4} = \frac{1}{a^4}$, $\therefore a^{-4} = \frac{1}{a^4}$.

Hence, if it be required to transfer a power in the numerator of a quantity into the denominator, or one in the denominator into the numerator, it may be effected by changing the sign of its index.

$$\frac{a}{b} = a . \frac{1}{b} = ab^{-1};\ \frac{a^m}{b^n} = a^m . \frac{1}{b^n} = a^m b^{-n}.$$

13. If m or n is negative, or if both are negative,

$$a^m . a^n = a^{m+n}.$$

(I.) Let both exponents be negative, then

$$a^{-m}=\frac{1}{a^m},\ a^{-n}=\frac{1}{a^n};$$

$$\therefore a^{-m}.\,a^{-n}=\frac{1}{a^m}\cdot\frac{1}{a^n}=\frac{1}{a^{m+n}}=a^{-m-n}=a^{(-m)+(-n)}.\quad \text{Q. E. D.}$$

(II.) Let now one of the exponents, m, be negative.

(1) If $m > n$, let $m=n+\Delta$;

$$\therefore a^{-m}=\frac{1}{a^{n+\Delta}}=\frac{1}{a^n.\,a^\Delta}\quad (3)$$

$$a^{-m}.\,a^n=\frac{a^n}{a^n.\,a^\Delta}=\frac{1}{a^\Delta}=a^{-\Delta}.$$

But $-\Delta=-n-\Delta+n=-(n+\Delta)+n=(-m)+n$;

$$\therefore a^{-m}.\,a^n=a^{(-m)+n}.\quad \text{Q. E. D.}$$

(2) If $m < n$, let $n=m+\Delta$.

Then $a^n=a^{m+\Delta}=a^m.\,a^\Delta$ (3); $a^{-m}.\,a^n=\frac{1}{a^m}.\,a^n=\frac{a^n}{a^m}=\frac{a^m.\,a^\Delta}{a^m}=a^\Delta$.

But $\Delta=-m+m+\Delta=-m+(m+\Delta)=(-m)+n$;

$$\therefore a^{-m}.\,a^n=a^{(-m)+n}.\quad \text{Q. E. D.}$$

(3) When $m=n$, $a^{-m}\,.\,a^n=a^{-m}\,.\,a^m=\frac{1}{a^m}.\,a^m=\frac{a^m}{a^m}=1=a^0=a^{(-m)+n}$.

14. For all positive and negative values of m and n,

$$a^m\div a^n=a^{m-n}.$$

For, by (13), $a^{m-n}.\,a^n=a^{m-n+n}$;

$\therefore$ by the principle of division, $a^m\div a^n=a^{m-n}$.

15. Again, $(abcd\ldots)^{-n}=a^{-n}b^{-n}c^{-n}d^{-n}\ldots$

For $(abcd\ldots)^{-n}=\frac{1}{(abcd..)^n}=\frac{1}{a^nb^nc^nd^n..(6)}=\frac{1}{a^n}\cdot\frac{1}{b^n}\cdot\frac{1}{c^n}\cdot\frac{1}{d^n}..=a^{-n}b^{-n}c^{-n}d^{-n}$.

16. $\left(\frac{a}{b}\right)^{-n}=\frac{a^{-n}}{b^{-n}}$.

$$\text{For}\quad \left(\frac{a}{b}\right)^{-n}=1\div\left(\frac{a}{b}\right)^n=1\div\frac{a^n}{b^n}\ (7)=1\,.\frac{b^n}{a^n}=\frac{b^n}{a^n};$$

$$\frac{a^{-n}}{b^{-n}}=a^{-n}\div b^{-n}=\frac{1}{a^n}\div\frac{1}{b^n}=\frac{1}{a^n}\cdot\frac{b^n}{1}=\frac{b^n}{a^n};$$

$$\therefore \left(\frac{a}{b}\right)^{-n}=\frac{a^{-n}}{b^{-n}}.\quad \text{Q. E. D.}$$

17. When m and n are both negative, or when only one of them is negative,

$$(a^m)^n=a^{mn}.$$

(I.) Let one of the exponents, m, be negative.

$$(1)\ (a^{-m})^n=\left(\frac{1}{a^m}\right)^n=\frac{1}{(a^m)^n}\ (7)=\frac{1}{a^{mn}}\ (8)=a^{-mn}=a^{(-m)n}.$$

(2) n being negative, $(a^m)^{-n}=\frac{1}{(a^m)^n}=\frac{1}{a^{mn}}$ (8)

(II.) Both the exponents being negative:

$$(a^{-m})^{-n}=\frac{1}{(a^{-m})^n}=\frac{1}{a^{-mn}}=1+a^{-mn}=1+\frac{1}{a^{mn}}=1\,.\,\frac{a^{mn}}{1}=a^{mn}=a^{(-m)(-n)}.$$

18. If the index of the root n is positive, and is contained in m, then

$$\sqrt[n]{a^m}=a^{\frac{m}{n}}.$$

For since m is divisible by n, then $\frac{m}{n}$ is either a positive or negative whole number, $\therefore$ by (17), $\left(a^{\frac{m}{n}}\right)^n=a^{n\left(\frac{m}{n}\right)}=a^m$; $\therefore$ (2), $a^{\frac{m}{n}}=\sqrt[n]{a^m}$.

19. Every power which occurs as a factor in the numerator or denominator of a fraction may be removed into the denominator or numerator respectively, without altering the value of the fraction.

$$\text{For } \frac{Aa^n}{B}=\frac{Aa^na^{-n}}{Ba^{-n}}=\frac{Aa^0}{Ba^{-n}}\,(13)=\frac{A}{Ba^{-n}}.$$

$$\frac{A}{Ba^n}=\frac{Aa^{-n}}{Ba^na^{-n}}=\frac{Aa^{-n}}{Ba^0}\,(13)=\frac{Aa^{-n}}{B}.$$

$$\frac{Aa^{-n}}{B}=\frac{Aa^{-n}a^n}{Ba^n}=\frac{Aa^0}{Ba^n}\,(13)=\frac{A}{Ba^n}.$$

$$\frac{A}{Ba^{-n}}=\frac{Aa^n}{Ba^{-n}a^n}=\frac{Aa^n}{Ba^0}\,(13)=\frac{Aa^n}{B}.$$

C. *Powers with Fractional Exponents.*

20. If n is positive, m positive or negative, and p positive, then

$$\sqrt[n]{a^m}=\sqrt[np]{a^{mp}}.$$

For $(\sqrt[np]{a^{mp}})^{np}=a^{mp}$ (2), $(\sqrt[n]{a^m})^{np}=((\sqrt[n]{a^m})^n)^p$ (8) $=(a^m)^p=a^{mp}$ (8) (17),

$$(\sqrt[n]{a^m})^{np}=(\sqrt[np]{a^{mp}})^{np},$$

$$\therefore\ \sqrt[n]{a^m}=\sqrt[np]{a^{mp}}.$$

21. If m and n are divisible by p, then

$$\sqrt[n]{a^m}=\sqrt[n+p]{a^{m\div p}}.$$

For $(n+p)p=n$, $(m+p)p=m$, $\therefore$ by (20), $\sqrt[n\div p]{a^{m\div p}}=\sqrt[(n\div p)p]{a^{(m+p)p}}=\sqrt[n]{a^m}$.

22. By what has preceded,

$$a^{\frac{m}{n}}=a^{\frac{mp}{np}}.$$

And also, if m and n are divided by p,

$$a^{\frac{m}{n}}=a^{\frac{m\div p}{n+p}}\ (20)\,(21).$$

23. $\sqrt[n]{abcde\ldots}=\sqrt[n]{a}\,.\,\sqrt[n]{b}\,.\,\sqrt[n]{c}\,.\,\sqrt[n]{d}\ldots$ (6) (2).

24. $\sqrt[n]{\frac{a}{b}}=\frac{\sqrt[n]{a}}{\sqrt[n]{b}}$ (7) (2).

25. $\sqrt[n]{a^m}\,.\,\sqrt[q]{a^p}=\sqrt[nq]{a^{mq+np}}$, where n and q are positive, and m and p may be positive or negative.

For by (20), $\sqrt[n]{a^m}=\sqrt[nq]{a^{mq}}$; $\sqrt[q]{a^p}=\sqrt[nq]{a^{np}}$;

$$\therefore\quad \sqrt[n]{a^m}\,.\,\sqrt[q]{a^p}=\sqrt[nq]{a^{mq}}\,.\,\sqrt[nq]{a^{np}}=\sqrt[nq]{a^{mq+np}}\quad (23)\,(13).$$

26. $a^{\frac{m}{n}}\,.\,a^{\frac{p}{q}}=a^{\frac{m}{n}+\frac{p}{q}}$.

For $a^{\frac{m}{n}}=\sqrt[n]{a^m}$, $a^{\frac{p}{q}}=\sqrt[q]{a^p}$;

$\therefore$ by (25), $a^{\frac{m}{n}}\,.\,a^{\frac{p}{q}}=\sqrt[n]{a^m}\,.\,\sqrt[q]{a^p}=\sqrt[nq]{a^{mq+np}}=a^{\frac{mq+np}{nq}}=a^{\frac{mq}{nq}+\frac{np}{nq}}=a^{\frac{m}{n}+\frac{p}{q}}$.

27. From this (26) it clearly follows that

$$a^{\frac{m}{n}}\,.\,a^{\frac{p}{q}}\,.\,a^{\frac{r}{s}}\,.\,a^{\frac{t}{u}}\ldots=a^{\frac{m}{n}+\frac{p}{q}+\frac{r}{s}+\frac{t}{u}\ldots}.$$

28. By means of the last propositions the multiplication of quantities with fractional indices is easily effected; *e.g.*

$$\sqrt[6]{a^5}\,.\,\sqrt[8]{a^{-7}}\,.\,\sqrt[12]{a^{-7}}\,.\,\sqrt[9]{a^5}\,.\,\sqrt[4]{a^3}$$
$$=a^{\frac{5}{6}}\;.\;a^{-\frac{7}{8}}\;.\;a^{-\frac{7}{12}}\;.\;a^{\frac{5}{9}}\;.\;a^{\frac{3}{4}},$$

and adding the indices we obtain $\frac{49}{72}$; and $\therefore$ the product of the quantities is $a^{\frac{49}{72}}=\sqrt[72]{a^{49}}$.

29. From (26) it follows that

$$a^{\frac{m}{n}}\div a^{\frac{p}{q}}=a^{\frac{m}{n}-\frac{p}{q}}.$$

Ex. $\sqrt[8]{a^3}\div\sqrt[4]{a^{-5}}=a^{\frac{3}{8}}\div a^{-\frac{5}{4}}=a^{\frac{3}{8}+\frac{5}{4}}=a^{\frac{13}{8}}=a^{1\frac{5}{8}}=a\,.\,a^{\frac{5}{8}}=a\sqrt[8]{a^5}$.

30. $(abcde\ldots)^{\frac{m}{n}}=a^{\frac{m}{n}}b^{\frac{m}{n}}c^{\frac{m}{n}}d^{\frac{m}{n}}\ldots$

For, by (7) (15) (23), $(abcde\ldots)^{\frac{m}{n}}=\sqrt[n]{(abcde\ldots)^m}=\sqrt[n]{(a^mb^mc^md^m\ldots)}=$

$$\sqrt[n]{a^m}\,.\,\sqrt[n]{b^m}\,.\,\sqrt[n]{c^m}\,.\,\sqrt[n]{d^m}\ldots=a^{\frac{m}{n}}b^{\frac{m}{n}}c^{\frac{m}{n}}d^{\frac{m}{n}}\ldots$$

31. $\left(\frac{a}{b}\right)^{\frac{m}{n}}=\frac{a^{\frac{m}{n}}}{b^{\frac{m}{n}}}$.

According to (8) (16) (24),

$$\left(\frac{a}{b}\right)^{\frac{m}{n}}=\sqrt[n]{\left(\frac{a}{b}\right)^m}=\sqrt[n]{\frac{a^m}{b^m}}=\frac{\sqrt[n]{a^m}}{\sqrt[n]{b^m}}=\frac{a^{\frac{m}{n}}}{b^{\frac{m}{n}}}.$$

32. If n and q are positive, but m and p either positive or negative,

$$\sqrt[q]{(\sqrt[n]{a^m})^p}=\sqrt[nq]{a^{mp}}.$$

For, by (2), $\sqrt[nq]{(a^{mp})^{nq}}=a^{mp}$;

and $(\sqrt[q]{(\sqrt[n]{a^m})^p})^{nq}=((\sqrt[q]{(\sqrt[n]{a^m})^p})^q)^n \; (8) =((\sqrt[n]{a^m})^p)^n \; (2) = (\sqrt[n]{a^m})^{pn}=$

$$((\sqrt[n]{a^m})^n)^p \; (9)\,(18) = (a^m)^p \; (2) = a^{mp} \; (8)\,(18);$$

$$\therefore \quad (\sqrt[nq]{a^{mp}})^{nq}=(\sqrt[q]{(\sqrt[n]{a^m})^p})^{nq};$$

and it follows also that $\sqrt[nq]{a^{mp}}=(\sqrt[q]{(\sqrt[n]{a^m})^p}$.

33. $(a^{\frac{m}{n}})^{\frac{p}{q}}=a^{\frac{mp}{nq}}$.

By (32), $(a^{\frac{m}{n}})^{\frac{p}{q}}=\sqrt[q]{(\sqrt[n]{a^m})^p}$; $a^{\frac{mp}{nq}}=\sqrt[nq]{a^{mp}}$; $\sqrt[q]{(\sqrt[n]{a^m})^p}=\sqrt[nq]{a^{mp}}$;

$$\therefore \quad (a^{\frac{m}{n}})^{\frac{p}{q}}=a^{\frac{mp}{nq}}.$$

34. Again, $\sqrt[p]{(a^{\frac{m}{n}})}=a^{\frac{m}{np}}$.

For, by (33), $\sqrt[p]{(a^{\frac{m}{n}})}=(a^{\frac{m}{n}})^{\frac{1}{p}}=a^{\frac{m}{np}}$.

It will be seen by 13, 14, 15, 16, 17, that the formulæ contained in section 11 hold for quantities with *negative* exponents. They are equally applicable to quantities with *fractional* exponents, and therefore they are general, and embrace the whole theory of indices.

We will here add a few examples for practice.

Exercise XXXIX.

Express with fractional indices:

1. $\sqrt{a}$, $\sqrt[5]{a^4}$, $(\sqrt{a})^3$, $(\sqrt[3]{a})^2$.
2. $\sqrt[3]{x^3yz^3}$, $\sqrt{a^3b^2c}$, $\sqrt[6]{xyz^5}$, $\sqrt[4]{a^3b^8}$.
3. $x\sqrt{y^3}$, $(\sqrt[3]{a})^5$, $\sqrt[4]{a^2b^3c}$, $\sqrt[3]{a^3bc^4}$.
4. $\sqrt[6]{x^3y^8}$, $x(\sqrt[4]{y})^3$, $\sqrt[5]{x^3y^{10}}$, $\sqrt{x^4y}$.

Transpose the powers (1) into the denominators, and (2) into the numerators:

5. x^{-1}, c^{-2}, b^{-4}, $\frac{2a}{c^2}$, $\frac{4x^2}{y^2}$, $\frac{5a^3}{c^3}$, $\frac{2x}{3y^2}$.
6. $\frac{x^3}{y^2z}$, $\frac{3x^2}{2yz}$, $\frac{3abc}{d}$, $\frac{bc^2}{2\sqrt{a}}$, $\frac{c}{3\sqrt[3]{a^2b}}$.

EXERCISE XL.

Express with sign of evolution :

1. $x^{\frac{1}{2}}$, $(3x)^{\frac{2}{3}}$, $(4a)^{\frac{3}{4}}$, $(7b)^{\frac{1}{2}}$, $y^{\frac{1}{4}}$.

2. $\frac{x^{\frac{1}{2}}}{y^{\frac{1}{2}}}$, $\frac{x^{\frac{1}{2}}y^{\frac{1}{3}}}{3a^{\frac{1}{2}}b^{\frac{1}{3}}}$, $\frac{4a^{\frac{1}{2}}b^{\frac{1}{3}}}{6c^{\frac{1}{2}}}$, $\frac{a^{\frac{2}{3}}b^{\frac{1}{2}}}{a^{\frac{1}{2}}b^{\frac{1}{3}}}$, $\frac{x^{\frac{1}{2}}y^{\frac{2}{3}}}{4a^{\frac{1}{2}}b^{\frac{1}{3}}}$.

EXERCISE XLI.

Express with sign of evolution, and with *positive* indices :

1. xy^{-1}, $a^{-2}bc^{-1}$, $x^{-1}y^{-1}z^{-2}$, $x^{2}y^{-1}z^{-3}$.

2. $x^{-\frac{1}{2}}$, $x^{\frac{1}{2}}y^{-\frac{1}{3}}$, $x^{-\frac{2}{3}}y^{\frac{1}{2}}$, $x^{\frac{2}{3}}y^{-\frac{1}{2}}$.

3. $\frac{x^{-1}yz^{-2}}{a^{-1}}$, $\frac{3x}{y^{-2}z^{-1}}$, $\frac{4bc^{-1}}{a^{-3}}$, $\frac{1}{x^{-1}y^{-2}z^{-3}}$.

4. $\frac{a^{-3}}{b^{-\frac{1}{2}}}$, $\frac{x^{-\frac{1}{2}}}{y^{-\frac{1}{3}}}$, $\frac{x^{-\frac{1}{3}}}{y^{-1}}$, $\frac{a^{-\frac{1}{2}}}{b^{-3}}$.

EXERCISE XLII.

Powers of the Powers of Quantities.

1. $(a^{\frac{m}{n}})^{\frac{p}{q}}$, $(a^{-\frac{m}{n}})^{\frac{p}{q}}$, $(a^{\frac{m}{n}})^{-\frac{p}{q}}$, $(a^{-\frac{m}{n}})^{-\frac{p}{q}}$.

2. $(a^{\frac{2}{3}}b^{\frac{3}{4}})^{\frac{1}{2}}$, $(a^{2}b^{-\frac{1}{3}}c^{\frac{2}{3}})^{-\frac{1}{4}}$.

3. $\{(a^{\frac{1}{2}})^{\frac{2}{3}}\}^{-\frac{1}{3}}$, $\sqrt[6]{(a^{3}b\sqrt[5]{a^{3}bc})^{5}}$.

4. $\left\{\frac{c^{2}d}{(a+b)^{\frac{2}{3}}}\right\}^{-\frac{1}{3}}$, $\sqrt[4]{\left(\frac{a\sqrt{b}}{\sqrt[3]{ab}}\right)^{3}}$.

5. $\frac{\sqrt{(c-d)}\,.\,\sqrt[3]{(a+x^{2})^{4}}}{c^{5}d^{5m}}$.

Ex. Multiplication :

$$\begin{array}{l}
x^{\frac{3}{4}}+x^{\frac{1}{2}}y^{\frac{1}{2}}+x^{\frac{1}{4}}y+y^{\frac{3}{2}} \\
x^{\frac{1}{4}}-y^{\frac{1}{2}} \\
\hline
x+x^{\frac{3}{4}}y^{\frac{1}{2}}+x^{\frac{1}{2}}y+x^{\frac{1}{4}}y^{\frac{3}{2}} \\
\quad -x^{\frac{3}{4}}y^{\frac{1}{2}}-x^{\frac{1}{2}}y-x^{\frac{1}{4}}y^{\frac{3}{2}}-y^{2} \\
\hline
x \quad . \quad . \quad . \quad -y^{2}
\end{array}$$

Ex. Division : $a^{3}-2\sqrt[4]{a^{2}b^{3}}-a^{2}\sqrt[6]{a^{3}b^{2}}+2b\sqrt[12]{b}\div(\sqrt{a}-\sqrt[3]{b})=$

$$\begin{array}{l}
a^{\frac{1}{2}}-b^{\frac{1}{3}})\,a^{3}-2a^{\frac{1}{2}}b^{\frac{3}{4}}-a^{\frac{5}{2}}b^{\frac{1}{3}}+2b^{\frac{13}{12}}\,(a^{\frac{5}{2}}-2b^{\frac{3}{4}} \\
\qquad\qquad a^{3} \qquad\qquad -a^{\frac{5}{2}}b^{\frac{1}{3}} \\
\qquad\qquad \overline{\quad -2a^{\frac{1}{2}}b^{\frac{3}{4}} \qquad\qquad +2b^{\frac{13}{12}}} \\
\qquad\qquad \quad -2a^{\frac{1}{2}}b^{\frac{3}{4}} \qquad\qquad +2b^{\frac{13}{12}} \\
\qquad\qquad \overline{\quad . \qquad . \qquad . \quad}
\end{array}$$

EXERCISE XLIII.

1. Simplify $2\sqrt[12]{\sqrt[5]{7}}+3\sqrt[6]{\sqrt[10]{7}}+3\sqrt[5]{\sqrt[12]{7}}-\sqrt[10]{\sqrt[6]{7}}$.
2. Simplify $\{(\frac{25}{49}a^2b^6c^8)^{\frac{1}{3}}\}^{\frac{1}{2}}$.
3. Simplify $2\sqrt{2\sqrt{2\sqrt{2\sqrt{2}}}}$.
4. Multiply $2a-3b^{\frac{1}{2}}+4c^{\frac{1}{3}}$ by $2a+3b^{\frac{1}{2}}+4c^{\frac{1}{3}}$.
5. Multiply $(a^{\frac{3}{4}}+b^{\frac{3}{4}})$ by $(a^{\frac{3}{4}}-b^{\frac{3}{4}})$, and $5a^{\frac{1}{4}}-6a^{\frac{3}{4}}b$ by $a^{\frac{1}{3}}-7a^{-\frac{2}{3}}b$.
6. Multiply $a^{-\frac{1}{2}}b^{-\frac{1}{3}}-2a^{-\frac{1}{4}}b^{\frac{3}{4}}c$ by $a^{\frac{1}{2}}-a^{-\frac{1}{4}}b$.
7. Multiply $\frac{(ad)^{\frac{1}{2}}b}{cf^{\frac{1}{2}}}-\frac{(ac)^{\frac{1}{3}}cd}{(bg)^{\frac{1}{2}}}$ by $\frac{(ad)^{\frac{1}{2}}b}{cf^{\frac{1}{2}}}+\frac{(ac)^{\frac{1}{3}}cd}{(bg)^{\frac{1}{2}}}$.
8. Divide $a^3-2a^{\frac{3}{2}}b^{\frac{3}{4}}-a^{\frac{3}{2}}b^{\frac{1}{2}}+2a^{\frac{1}{2}}b^{\frac{5}{4}}$ by $a^{\frac{3}{2}}-b^{\frac{1}{2}}$.
9. Divide $a^{\frac{1}{2}}b^{\frac{1}{3}}$ by $a^{-\frac{1}{2}}b^{-\frac{1}{4}}c$, and $ca^{\frac{3}{4}}$ by $da^{\frac{1}{3}}$.
10. Divide $a^{\frac{1}{2}}b^{\frac{3}{4}}-ca^{\frac{7}{10}}b^{\frac{1}{2}}-\frac{3}{2}ab^{\frac{3}{4}}+\frac{3}{2}ca^{\frac{11}{10}}b^{\frac{1}{2}}$ by $a^{\frac{1}{2}}b^{\frac{1}{4}}-\frac{3}{2}a^{\frac{3}{2}}b^{\frac{1}{4}}$.
11. Divide $5a^{\frac{11}{6}}-41a^{\frac{11}{12}}b+42a^{\frac{1}{12}}b^2$ by $a^{\frac{2}{3}}-7ba^{-\frac{1}{4}}$.
12. Divide $a^{\frac{3}{2}}-b^{\frac{3}{2}}$ by $a^{\frac{1}{2}}-b^{\frac{1}{2}}$.

CHAPTER XI.

SURDS.

NUMERICAL and algebraic expressions of the form $\sqrt{3}$, $\sqrt[3]{9}$, $\sqrt[4]{18}$, $\sqrt{1+x^2}$, $\sqrt[3]{x^3+x^2y}$, $\sqrt[4]{3x^4}$, whose root cannot be exactly determined, are called irrational quantities, or surds.

These quantities may be expressed by employing fractional indices, thus: $3^{\frac{1}{2}}$; $9^{\frac{1}{3}}$; $18^{\frac{1}{4}}$; $(1+x^2)^{\frac{1}{2}}$; $(x^3+x^2y)^{\frac{1}{3}}$; $(3x^4)^{\frac{1}{4}}$.

We have seen that a negative number can have no square root, and that no number multiplied by itself will result in a negative product. Such a term is called an *imaginary* or *impossible* quantity; *e.g.* $\sqrt{-16}$, $\sqrt{-y^2}$. Thus, what shall we assign for the $\sqrt{-16}$? It cannot be $+4$ or -4, for the square of each is $+16$; it cannot, in fact, be any number positive or negative. Hence the square root, or any *even* root, of every negative quantity cannot be expressed by any species of quantity with which we are acquainted. When, therefore, we write down an impossible quantity, $\sqrt{-16}$, $\sqrt{-y^2}$, we simply declare that the square root of -16 and $-y^2$ is no possible number. When we square one of these quantities, it is evident we obtain the negative quantity of which we attempted to take the root, $\sqrt{-16}\times\sqrt{-16}=-16$, $\sqrt{-y^2}\times\sqrt{-y^2}=-y^2$.

Though these roots cannot be exactly determined, it will be readily perceived that the operations of addition, subtraction, multiplication, and division may be performed with them under their *surd* form.

It is frequently required to put a rational quantity in the form of a surd. Thus,

$$x=\sqrt{x^2}=\sqrt[3]{x^3},\quad 3=\sqrt{9}=\sqrt[3]{27}.$$

We see, from these examples, that in order to put a rational quantity in the form of a surd, we must raise the given quantity to a power whose index is the number which indicates the root, and place the sign of the root over it. Similarly a quantity partly rational and partly surd may be placed under the form of a complete surd, by raising the rational part to the power whose index is the number which denotes the root, and placing beneath the sign the product of this power and the surd part.

Ex. $4\sqrt{2}=\sqrt{16}\times\sqrt{2}=\sqrt{32}$; $2\sqrt[3]{5^2}=\sqrt[3]{8}\times\sqrt[3]{25}=\sqrt[3]{200}$;

$$2x\sqrt[3]{\frac{y}{2x}}=\sqrt[3]{\frac{8x^3y}{2x}}=\sqrt[3]{4x^2y}.$$

And, conversely, a quantity under the form of a surd may frequently be resolved into one partly rational and partly surd, by separating the quantity below the sign into such factors, that the root of one of them may be obtained, which will be the rational part.

Ex. $\sqrt{32}=\sqrt{16\times 2}=4\sqrt{2}$; $\sqrt[3]{200}=\sqrt[3]{8\times 25}=2\sqrt[3]{25}$;

$$\sqrt{\frac{27}{16}x^3y}=\sqrt{\frac{9x^2\times 3xy}{16}}=\frac{3}{4}x\sqrt{3xy}.$$

When the quantity below the root is reduced as low as possible, the surd is said to be in its most rational terms.

Ex. $\sqrt{45}=\sqrt{9}\times\sqrt{5}=3\sqrt{5}$; $\sqrt[3]{128}=\sqrt[3]{64}\times\sqrt[3]{2}=4\sqrt[3]{2}$;

$$\sqrt[4]{8^3}=\sqrt[4]{512}=\sqrt[4]{256}\times\sqrt[4]{2}=4\sqrt[4]{2};$$

$$5\sqrt[3]{\tfrac{4}{5}}=5\sqrt[3]{\frac{4\,.\,5^2}{5\,.\,5^2}}=5\sqrt[3]{\frac{100}{5^3}}=\frac{5\sqrt[3]{100}}{5}=\sqrt[3]{100}.$$

EXERCISE XLIV.

Express in the form of surds:

1. $3^{\frac{3}{4}}$, $4^{\frac{2}{3}}$, $5^{-\frac{3}{4}}$, $(\frac{2}{3})^{-\frac{3}{4}}$, with indices whose numerator is 1.
2. 5, 6, 7, 8, with indices $\frac{1}{2}$ and $\frac{1}{3}$.
3. $\frac{x+y}{3}$, with the index $\frac{1}{3}$.
4. $5x^2y$ and $\frac{1}{2}(a+b)$, with index $\frac{1}{4}$.

Place the following under the form of complete surds:

5. $2\sqrt{3}$; $3\sqrt{\frac{2}{3}}$; $\frac{3}{4}\,.\,2^{\frac{1}{2}}$; $\frac{4}{5}\sqrt{2\frac{1}{2}}$.
6. $\frac{1}{2}(\frac{3}{10})^{-\frac{1}{2}}$; $4\sqrt[3]{4}$; $6\,.\,3^{-\frac{1}{3}}$; $5\,.\,5^{\frac{1}{4}}$.
7. $2\,.\,4^{-\frac{1}{4}}$; $\frac{2}{3}(\frac{3}{4})^{-\frac{1}{2}}$; $4\sqrt{x}$; $3x\sqrt{3y}$; $x(2xy)^{-1}$; $(a-b)(a^2-b^2)^{-\frac{1}{2}}$.

Reduce to their simplest form:

8. $\sqrt{54}$; $\sqrt{75}$; $\sqrt{96}$; $\sqrt{128}$.
9. $9\sqrt{48}$; $\sqrt[3]{128}$; $\sqrt[3]{686}$; $4\sqrt[3]{54}$.
10. $2\sqrt[3]{448}$; $\sqrt{\frac{1}{2}}$; $2\sqrt{\frac{3}{2}}$; $5\sqrt[3]{\frac{2}{3}}$.
11. $9^{\frac{2}{3}}$; $12^{\frac{1}{2}}$; $(6\frac{1}{4})^{\frac{1}{3}}$; $(3\frac{1}{8})^{-\frac{1}{2}}$.

To add or subtract Surd quantities.

Rule. Reduce them to similar surds, and add or subtract their coefficients.

Similar surds are those which have the same quantity under the same radical sign. Thus

$$9\sqrt[3]{3} \text{ and } 5\sqrt[3]{3};\ 3x\sqrt{y} \text{ and } 5x\sqrt{y}.$$

Ex. Find the sum and difference of $\sqrt{243}$ and $\sqrt{27}$.

$$\sqrt{243}=\sqrt{81\times 3}=9\sqrt{3}, \text{ and } \sqrt{27}=\sqrt{9\times 3}=3\sqrt{3};$$

$\therefore$ adding coefficients of $9\sqrt{3}$ and $3\sqrt{3}$, we have $12\sqrt{3}$;

and subtracting, we have $6\sqrt{3}$.

EXERCISE XLV.

1. $b\sqrt[m]{a}+c\sqrt[m]{a}-d\sqrt[m]{a}$; $3\sqrt[6]{5}+17\sqrt[6]{5}-12\sqrt[6]{5}-7\sqrt[6]{5}$.
2. $\{18\sqrt{7}-5\sqrt{6}+10\sqrt[4]{11}-3\sqrt[3]{13}\}-\{6\sqrt{7}-2\sqrt{6}+\frac{17}{2}\sqrt[4]{11}+21\sqrt[3]{3}\}$.
3. $\sqrt{24}+\sqrt{54}-\sqrt{6}$; $\sqrt{8}-7\sqrt{18}+5\sqrt{72}-\sqrt{50}$.
4. $7\sqrt[3]{54}+3\sqrt[3]{16}+\sqrt[3]{2}-5\sqrt[3]{128}$.
5. $2\sqrt{\frac{5}{3}}+\sqrt{60}-\sqrt{15}+\sqrt{\frac{3}{5}}$.
6. $\sqrt{12}+2\sqrt{27}+3\sqrt{75}-9\sqrt{48}$.
7. $8\sqrt{\frac{3}{4}}-\frac{1}{2}\sqrt{12}+4\sqrt{27}-2\sqrt{\frac{3}{16}}$.
8. $3\sqrt[3]{81}-2\sqrt[3]{24}+\sqrt{28}+2\sqrt{63}$.
9. $12\sqrt[3]{\frac{1}{4}}+3\sqrt{\frac{1}{32}}$.
10. $\sqrt{45c^3}-\sqrt{80c^3}+\sqrt{5a^2c}$.

If the quantities cannot be reduced to similar surds, their sum or difference is expressed as in simple unlike quantities, by placing the proper sign between them. Thus, to add $6\sqrt{x}$ and $9\sqrt{y}$, we write $6\sqrt{x}+9\sqrt{y}$; to subtract $4\sqrt{b}$ from $7\sqrt{c}$, we put $7\sqrt{c}-4\sqrt{b}$.

To multiply and divide Surd quantities.

Rule. (1.) If they are surds of the same dimension—*i. e.* have the same radical sign—find the product or quotient of the surd parts; this, joined to the product or quotient of the rational parts, with the common radical sign between them, will give the product or quotient required.

(2.) If they are not of the same dimension, make the surd indices equal, and proceed as in (1.).

(1.) Ex. $\sqrt{7}\times\sqrt{6}=\sqrt{42}$; $\sqrt[3]{27}\div\sqrt[3]{6}=\sqrt[3]{\frac{27}{6}}=3\sqrt[3]{\frac{1}{6}}$;

$$\sqrt[4]{x^3y}\div\sqrt[4]{xy^3}=\sqrt[4]{\frac{x^3y}{xy^3}}=\sqrt[4]{\frac{x^2}{y^2}}=\sqrt{\frac{x}{y}}.$$

(2.) Ex. $\sqrt{x}\times\sqrt[3]{y}=x^{\frac{1}{2}}\times y^{\frac{1}{3}}=x^{\frac{3}{6}}\times y^{\frac{2}{6}}=\sqrt[6]{x^3}\times\sqrt[6]{y^2}=\sqrt[6]{x^3y^2}\cdot$

$$\sqrt[m]{x}\times\sqrt[n]{x}=x^{\frac{1}{m}}\times x^{\frac{1}{n}}=x^{\frac{n}{mn}}\times x^{\frac{m}{mn}}=x^{\frac{m+n}{mn}}=\sqrt[mn]{x^{m+n}};$$

$$\sqrt{c}\div\sqrt[3]{c}=\frac{c^{\frac{1}{2}}}{c^{\frac{1}{3}}}=\frac{c^{\frac{3}{6}}}{c^{\frac{2}{6}}}=c^{\frac{1}{6}}=\sqrt[6]{c}.$$

Exercise XLVI.

1. $a\sqrt[n]{x}\times b\sqrt[n]{x}\times c\sqrt[n]{x}$; $4\times2\sqrt[6]{3}\times\sqrt[6]{72}$.
2. $5\sqrt{3}\times7\sqrt{\frac{8}{3}}\times\sqrt{2}$; $c\sqrt{a}\times d\sqrt{a}$.
3. $\sqrt[m]{a}\times\sqrt[n]{b}$; $\sqrt{2}\times\sqrt[3]{3}\times\sqrt[4]{5}$.
4. $\sqrt[3]{4}\times7\sqrt[3]{6}\times\frac{1}{2}\sqrt[3]{5}$; $(\sqrt{5}+2\sqrt{7}+3\sqrt{10})\times2\sqrt{5}$.
5. $(2\sqrt{6}-3\sqrt{5})\times4\sqrt{3}-\sqrt{10}$; $(\sqrt{2}+\sqrt{3})\times(2\sqrt{2}-\sqrt{3})$.
6. $\sqrt[m]{a}\div\sqrt[m]{b}$; $c\sqrt[m]{a}\div d\sqrt[m]{b}$; $a\div\sqrt[m]{b}$.
7. $a\div\sqrt{a}$; $2ab^2c^3\div4\sqrt[3]{a^3bc^5d}$.
8. $4\sqrt[3]{12}\div2\sqrt{3}$; $(\sqrt{72}+\sqrt{32}-4)\div\sqrt{8}$.
9. $(\sqrt{6}+4\sqrt{18}-3-8\sqrt{2})\div\sqrt{3}$.
10. $(\sqrt{6}+2\sqrt{3}-\sqrt[3]{18})\div\sqrt{6}$.

To rationalise a given Binomial Surd; as $a+\sqrt{b}$, *or* $\sqrt{a}+\sqrt{b}$.

$$x^2-y^2=(x+y)(x-y).$$

If $x=a$, and $y=\sqrt{b}$, then $a^2-b=(a+\sqrt{b})(a-\sqrt{b})$; therefore, to rationalise $a+\sqrt{b}$, we must multiply it by the same surd with the sign changed.

If $x=\sqrt{a}$, and $y=\sqrt{b}$, then $a-b=(\sqrt{a}+\sqrt{b})(\sqrt{a}-\sqrt{b})$; $\sqrt{a}+\sqrt{b}$ will therefore be rendered rational by multiplying it by $\sqrt{a}-\sqrt{b}$.

Any surd, therefore, of the form $a\pm\sqrt{b}$, or $\sqrt{a}\pm\sqrt{b}$, may be made rational by being multiplied by a similar surd with the sign changed.

Ex. Let $5+\sqrt{3}$ and $\sqrt{6}-\sqrt{3}$ be the surds to be rendered rational. The multipliers will be $5-\sqrt{3}$ and $\sqrt{6}+\sqrt{3}$, and the products will be $25-3$, or 22, and $6-3$, or 3.

If the surd is of the form $\sqrt[3]{a}+\sqrt[3]{b}$, how shall it be rendered rational?

Now, $(a^{\frac{1}{3}}+b^{\frac{1}{3}})(a^{\frac{2}{3}}-a^{\frac{1}{3}}b^{\frac{1}{3}}+b^{\frac{2}{3}})=a+b$; $a^{\frac{2}{3}}-a^{\frac{1}{3}}b^{\frac{1}{3}}+b^{\frac{2}{3}}$ will be the multiplier which will render $\sqrt[3]{a}+\sqrt[3]{b}$ rational.

Ex. What multiplier will render $\sqrt[3]{4}+2\sqrt[3]{3}$ rational?

$$a=4,\ b=24;\ \therefore\ \sqrt[3]{4}+2\sqrt[3]{3}=\sqrt[3]{4}+\sqrt[3]{24}.$$

$\therefore\ 4^{\frac{2}{3}}-4^{\frac{1}{3}}.24^{\frac{1}{3}}+24^{\frac{2}{3}}$ is the rationalising factor, and the product will be 28.

EXERCISE XLVII.

Rationalise the denominators of

1. $\dfrac{1}{(2+\sqrt{3})};\ \dfrac{3}{(1+\sqrt{2})};\ \dfrac{12}{(5-\sqrt{21})}.$

2. $\dfrac{7}{(\sqrt{8}-2)};\ \dfrac{\sqrt{3}}{(2\sqrt{5}-3\sqrt{2})}.$

Simplify

3. $\dfrac{\frac{1}{2}\sqrt{\frac{1}{2}}}{(\sqrt{2}+3\sqrt{\frac{1}{2}})};\ \dfrac{(3\sqrt{5}-2\sqrt{2})}{(2\sqrt{5}-\sqrt{18})}.$

4. $\dfrac{\sqrt[4]{a}+\sqrt[4]{b}}{\sqrt[4]{a}-\sqrt[4]{b}}.$

To extract the square root of a binomial quadratic surd, as $a+\sqrt{b}$.

First let us demonstrate:

1. That a surd quantity cannot be equal to one partly rational and partly a quadratic surd.

$$\text{Let } \sqrt{x}=a+\sqrt{b},\ \text{then } x=a^2+2a\sqrt{b}+b,$$

$$\therefore\ 2a\sqrt{b}=x-(a^2+b),\quad \therefore\ \sqrt{b}=\frac{x-(a^2+b)}{2a};$$

that is, a surd equals a rational quantity, which is contrary to the definition of a surd.

2. A surd cannot equal the sum or difference of two dissimilar surds.

$$\begin{aligned} \text{Let}\quad & \sqrt{x}=\sqrt{a}\pm\sqrt{b}, \\ \text{squaring,}\quad & x=a\pm2\sqrt{ab}+b, \\ & \pm2\sqrt{ab}=(a+b-x), \\ & \pm\sqrt{ab}=\tfrac{1}{2}(a+b-x), \end{aligned}$$

that is, a surd quantity is equal to a rational quantity, which is impossible.

3. If there be an equation between rational quantities and surds, as $a+\sqrt{b}=x+\sqrt{y}$, then $a=x$, and $\sqrt{b}=\sqrt{y}$.

If a is not equal to x, let it be greater than it by p, then $a=x+p$, and $x+p+\sqrt{b}=x+\sqrt{y}$, $\therefore\ p+\sqrt{b}=\sqrt{y}$; that is to say, a quantity partly rational and partly a quadratic surd is equal to a quadratic surd, which is impossible, by (1).

4. The product of two dissimilar surds is a surd.

Let the product of $\sqrt{x}$ and $\sqrt{y}$ be a rational quantity, say px, then $p^2x^2=xy$, $\therefore$ $p^2x=y$, $\therefore$ $p\sqrt{x}=\sqrt{y}$; that is, $\sqrt{x}$ and $\sqrt{y}$ involve the same surd, which is contrary to the supposition of their being dissimilar. Therefore $\sqrt{x}\times\sqrt{y}$, or $\sqrt{xy}$, is a surd.

Now, to extract the square root of a binomial surd. The formula is:

$$\sqrt{a\pm\sqrt{b}}=\sqrt{\frac{a+\sqrt{a^2-b}}{2}}\pm\sqrt{\frac{a-\sqrt{a^2-b}}{2}}.$$

To prove this, suppose:

$$\sqrt{a+\sqrt{b}}=\sqrt{x}+\sqrt{y},$$

$$\text{or}\quad a+\sqrt{b}=x+2\sqrt{xy}+y\,;$$

$$\therefore\quad a=x+y,\ \text{and}\ \sqrt{b}=2\sqrt{xy},\ \text{and by squaring,}$$

$$a^2=x^2+2xy+y^2,\ b=4xy,$$

$$\therefore\quad a^2-b=x^2-2xy+y^2=(x-y)^2,$$

$$\therefore\quad x-y=\sqrt{a^2-b}=p.$$

By adding and subtracting the two equations, $x+y=a$, $x-y=p$,

we have $\qquad x=\frac{a+p}{2}$, and $y=\frac{a-p}{2}$,

$$\therefore\ \sqrt{a\pm\sqrt{b}}=\sqrt{\frac{a+p}{2}}\pm\sqrt{\frac{a-p}{2}}=\sqrt{\frac{a+\sqrt{a^2-b}}{2}}\pm\sqrt{\frac{a-\sqrt{a^2-b}}{2}}.$$

If $\sqrt{a+\sqrt{b}}=\sqrt{x}+\sqrt{y}$, then $\sqrt{a-\sqrt{b}}=\sqrt{x}-\sqrt{y}$.

We have seen that $a=x+y$, and $\sqrt{b}=2\sqrt{xy}$; subtracting,

$$a-\sqrt{b}=x-2\sqrt{xy}+y=(\sqrt{x}-\sqrt{y})^2,$$

$$\therefore\quad \sqrt{a-\sqrt{b}}=\sqrt{x}-\sqrt{y}.$$

So that binomial surds of the form $\sqrt{a\pm\sqrt{b}}$ must be equated with $\sqrt{x}\pm\sqrt{y}$, and put through a process similar to the above.

Exercise XLVIII.

1. $\sqrt{(7+4\sqrt{3})}$.
2. $\sqrt{(43-15\sqrt{8})}$.
3. $\sqrt{(5-\sqrt{24})}$.
4. $\sqrt{(3-2\sqrt{2})}$.
5. $\sqrt{(87-12\sqrt{42})}$.
6. $\sqrt{(\frac{3}{2}+\sqrt{2})}$.
7. $\sqrt{(2+\sqrt{3})}$.
8. $\sqrt{(28+5\sqrt{12})}$.
9. $\sqrt{(\sqrt{32}-\sqrt{24})}$.
10. $\sqrt{(3\sqrt{5}+\sqrt{40})}$.
11. $\sqrt{(\sqrt{18}-4)}$.
12. $\{x-2\sqrt{(x-1)}\}$.
13. $\sqrt{\{2a+2\sqrt{(a^2-b^2)}\}}$.
14. $\sqrt{\left\{\frac{a^2}{4}+\frac{c}{2}\sqrt{(a^2-c^2)}\right\}}$.
15. $\sqrt{(x+xy-2x\sqrt{y})}$.
16. $\sqrt{\{ap-2a\sqrt{(ap-a^2)}\}}$.
17. $\sqrt{\left\{\frac{3a}{b}+\sqrt{\left(\frac{12a^3c^2}{bd^2}-\frac{4a^4c^4}{d^4}\right)}\right\}}$.
18. $\sqrt{\{b^2-ab+\frac{a^2}{4}+\sqrt{(4ab^3-8a^2b^2+a^3b)}\}}$.

Miscellaneous Examples.

Reduce to their simplest form :

1. $\sqrt{\frac{7}{25}}$, $\sqrt{\frac{28}{45}}$, $\sqrt[3]{\frac{5}{27}}$, $8\sqrt{\frac{3}{4}}-\frac{1}{2}\sqrt{12}+4\sqrt{27}-2\sqrt{\frac{3}{16}}$.

2. $\sqrt[4]{\left(\frac{16a^5x^7+16a^6x^6}{81b^4c^2-162b^5}\right)}$, $\sqrt[5]{\left(\frac{32a^9+128a^6b^3-160a^5b^4}{3b^6}\right)}$.

3. $3b^2\sqrt{a^3c}+\frac{2}{c}\sqrt{a^5c^3}-c^4\sqrt{\frac{ac}{b^2}}$, $5a\sqrt[3]{\frac{a^2}{b}}+b\sqrt{\frac{b^2c^3}{a}}$, $\sqrt[3]{54a^{m+6}b^3}-\sqrt[3]{16a^{m-3}b^6}+\sqrt[3]{2a^{4m+9}}+\sqrt[3]{2c^3a^m}$.

4. Reduce $\sqrt{(a^2c+a^2d)}$, $\sqrt[3]{(a^{6m}b-a^{7m}f^2)}$.

5. $\frac{a-b}{a+b}\sqrt{\frac{ac}{a^2-2ab+b^2}}$, $\sqrt{\left(\frac{x^3+2x^2+x}{a^3+a^2b}\right)}$.

6. Add together $\sqrt{(ay^2+by^2+cy^2)}$ and $\sqrt{(az^2+bz^2+cz^2)}$.

Add together $\sqrt{(a^2x+4ax+4x)}$, $\sqrt{(x^3+10x^2+25x)}$, and $\sqrt{(xy^2-8xy+16x)}$.

7. What is the difference between :

$\sqrt[4]{(b^2-c^2)}$ and $(a+3)\sqrt[4]{(81b^2-81c^2)}$,

$3a^2\sqrt[5]{a^8b^2}+5\sqrt[4]{a^5b^6}$ and $(b+c)\sqrt[5]{ab^2}$,

$2\sqrt{75}$ and $5\sqrt{2\frac{1}{12}}$, $3\sqrt{\frac{3}{4}}$ and $\frac{1}{2}\sqrt[3]{6}$.

8. Multiply $(\sqrt[3]{5}-2\sqrt[3]{6})$ by $(3\sqrt[3]{4}-\sqrt[3]{36})$; $(2p+q)\sqrt[3]{\frac{a-c}{a+c}}$ by $\sqrt[4]{(a^2-c^2)}$.

9. Multiply $(5+\sqrt[3]{4}+2\sqrt[4]{5})$ by $\sqrt{6}+\sqrt{5}$, $(\sqrt{a}+c\sqrt[3]{b})$ by $(\sqrt{a}-c\sqrt[3]{b})$, $(2\sqrt{a}+3c\sqrt[3]{b})$ by $(\sqrt{a}+4\sqrt[3]{b})$.

10. Divide $8\sqrt{10pq}$ by $4\sqrt{5q^3}$, $\sqrt[3]{(p^3+q^3)}$ by $c\sqrt[3]{(p^2-q^2)}$, $2\sqrt[4]{\frac{80a^3b^2}{c^3}}$ by $3\sqrt[4]{\frac{5c^2-cd}{162b^3d^2}}$, $(a^2-b^2)\sqrt[5]{\frac{2c}{3d^4}}$ by $a+b\sqrt[5]{c^2}$, $a\sqrt[m+1]{\frac{a^{m-3}}{b^n}}$ by $b+c\sqrt[m+1]{\frac{b^3}{a^4}}$.

11. Divide $a^2+2ab+b^2-x^2y$ by $a+b-x\sqrt{y}$, $xy-x^2+y^2$ by $\sqrt{xy}-\sqrt{(x^2-y^2)}$, $3\sqrt{15}-\sqrt{20}+\sqrt{10}-7$ by $2\sqrt{5}$.

12. Rationalise denominators $\frac{9\sqrt{7}}{4+\sqrt{7}}$, $\frac{4}{\sqrt{5}+\sqrt{3}}$, $\frac{12}{\sqrt{11}-\sqrt{2}}$, $\frac{8\sqrt{3}}{\sqrt{7}+\sqrt{3}}$, $\frac{\sqrt{15}}{\sqrt{10}-\sqrt{5}}$, $\frac{7+\sqrt{50}}{\sqrt{12}+\sqrt{6}}$, $\frac{18+8\sqrt{91}}{3\sqrt{13}-\sqrt{7}}$, $\frac{2\sqrt{3}-\sqrt{6}}{4-\sqrt{2}+\sqrt{6}}$.

13. Expand $(2+\sqrt{3})^2$, $(\frac{1}{2}\sqrt{7}-\frac{1}{2}\sqrt{3})^2$, $(a^2\sqrt{3c}+b^2\sqrt{3d})^2$, $(a^3-3b\sqrt[3]{c^2})^2$, $\{p^2+3(p+q)\sqrt{(p+q)}\}^2$, $(b-\frac{3b}{2c}\sqrt{ab})^3$, $(2\frac{1}{2}-4\sqrt[3]{5})^5$.

14. Place the following under a single radical sign : $\sqrt[3]{(x^2z\sqrt{xyz})}$, $\sqrt[4]{(b^2cd^3\sqrt[3]{a^2cd})}$, $\sqrt[3]{(16\sqrt{2a})}$, $\sqrt{\{2\sqrt[3]{(4\sqrt[5]{2b})}\}}$, $\sqrt{\{\sqrt{(4a^2+20ab+35b^2)}\}}$, $\sqrt[3]{\{\sqrt[5]{(a^3+6a^2b+12ab^2+8b^3)}\}}$.

15. Take the roots: $\sqrt{(67+\sqrt{4320})}$, $\sqrt{\{2a-2\sqrt{(a^2-b^2)}\}}$, $\sqrt{(3\sqrt{6}+2\sqrt{12})}$, $\sqrt{(x+xy-2x\sqrt{y})}$, $\sqrt{(6+\sqrt{31})}+\sqrt{(6-\sqrt{31})}$, $\sqrt{(24\frac{1}{2}+3\sqrt{66})}+\sqrt{(24\frac{1}{2}-3\sqrt{66})}$.

CHAPTER XII.

QUADRATIC EQUATIONS.

QUADRATIC equations are those which involve the *square* of the unknown quantity.

They are of two kinds:

1. *Pure* Quadratics of the form $x^2-16=0$, in which the simple power of the unknown quantity does not appear.

2. *Adfected* Quadratics of the form $ax^2+bx-c=0$, in which the first power of the unknown quantity is involved as well as its square.

All the rules hitherto given for the simplification of equations apply equally to Quadratics; indeed, it is not until a simplification has been effected that we can know whether the equation is *simple* or *quadratic*. Should the equation when reduced assume either of the above forms, it is a quadratic, and would be thus solved:

$$\begin{aligned}(1.)\quad x^2-16&=0,\\ x^2&=16,\\ \sqrt{x^2}&=\sqrt{16},\\ x&=\pm 4,\end{aligned}$$

or x has two values, $+4$ and -4; for either of these when put for x will satisfy the equation that $x^2-16=0$.

EXERCISE XLIX.

1. $5x^2-45=0.$

2. $8x^2=185+15.$

3. $ax^2-b=c.$

4. $x^2+\frac{3}{4}=\frac{259}{4}.$

5. $\frac{9x^2+4}{4}=10.$

6. $(x+2)^2=4x+5.$

7. $x\sqrt{6+x^2}=1+x^2.$

8. $\frac{x^2-12}{3}=\frac{x^2-4}{4}.$

9. $\frac{x+7}{x^2-7x}-\frac{x-7}{x^2+7x}=\frac{7}{x^2-7^3}.$

10. $ax^2-5c=bx^2-3c+d.$

But should the equation when simplified be of the form $ax^2+bx-c=0$, it is by the definition a quadratic, and its solution is not so obvious.

$$\begin{aligned}(2.)\qquad ax^2+bx-c&=0,\\ ax^2+bx&=c;\\ \text{dividing by } a,\quad x^2+\frac{b}{a}x&=\frac{c}{a};\end{aligned}$$

and putting p for $\frac{b}{a}$, and q for $\frac{c}{a}$, $x^2+px=q$. Now, if we can add a quantity r to both sides of the equation, so that x^2+px+r may be a complete square, we may reduce the equation to a simple one by taking the roots.

To find r the quantity to be added in order to make x^2+px a complete square.

We know that in the square of a binomial $x^2+2xy+y^2$ (the square of $x+y$) four times the product of the first and last terms are equal to the square of the middle term. Therefore in x^2+px+r,

$$4x^2r=p^2x^2,$$

$$\therefore\ r=\frac{p^2x^2}{4x^2}=\frac{p^2}{4}=\left(\frac{p}{2}\right)^2;$$

so that if we add $\left(\frac{p}{2}\right)^2$, the first side becomes a complete square.

$$x^2+px+\left(\frac{p}{2}\right)^2=q+\left(\frac{p}{2}\right)^2,$$

and taking the square root

$$x+\frac{p}{2}=\pm\sqrt{q+\frac{p^2}{4}},\ \text{and}\ x=-\frac{p}{2}\pm\sqrt{q+\frac{p^2}{4}},$$

the double sign being prefixed as before.

We may here remark that every equation has as many roots as it has dimensions. A quadratic therefore has two roots.

These roots are either *real* and *different*, *equal* or *impossible*, according as

$$-\frac{p^2}{4}\ \substack{<\\=\\>}\ q.$$

Hence we obtain the following rules for the solution of adfected quadratics:

1. Reduce the equation to its simplest form.
2. Transpose the terms involving the unknown quantities to one side of the equation, and the known quantities to the other.
3. To reduce the equation to the form $x^2+px-q=0$, divide every term of the equation by the coefficient of x^2.
4. Complete the square by adding to both sides of the equation the square of half the coefficient of x.
5. Take the square root of both sides, and, prefixing $\pm$ to that on the right, we obtain the *two* roots of the equation.

Ex. 1. $x^2+6x=27.$

$$x^2+6x+9=36.$$
$$x+3=\pm 6.$$
$$x=3 \text{ or } -9.$$

Ex. 2. $x^2+20=12x.$

$$\therefore\ x^2-12x=-20.$$
$$x^2-12x+(6)^2=-20+36=16.$$
$$x-6=\pm 4.$$
$$x=10 \text{ or } 2.$$

EXERCISE L.

1. $x^2-2x=24.$
2. $x^2-6x=7.$
3. $x^2+2x-35=0.$
4. $x^2-8x=20.$
5. $x^2+60x=700.$
6. $x^2-8x=-7.$
7. $x^2-7x+3\frac{1}{4}=0.$
8. $x^2-5\frac{3}{4}x=18.$
9. $3x^2-2x=65.$
10. $20748-1616x+21x^2=0.$
11. $9\frac{3}{8}x-21\frac{15}{16}=x^2.$
12. $11\frac{3}{4}x-3\frac{1}{2}x^2=-41\frac{1}{4}.$

EXERCISE LI.

1. $\frac{18x^2}{5}+\frac{18078x}{65}+4728=0.$

2. $x^2-8x=14.$

3. $3x^2+x=7.$

4. $118x-2\frac{1}{2}x^2=20.$

5. $80x+\frac{3x^2}{4}+\frac{21x-27782}{12}=1859\frac{1}{8}-3x^2.$

6. $\frac{x}{x+60}=\frac{7}{3x-5}.$

7. $\frac{8x}{x+2}-6=\frac{20}{3x}.$

8. $\frac{31}{6x}=\frac{16}{117-2x}+1.$

9. $\frac{x}{x+1}+\frac{x+1}{x}=\frac{13}{6}.$

10. $\frac{18+x}{6(3-x)}=\frac{20x+9}{19-7x}-\frac{65}{4(3-x)}.$

11. $adx-acx^2=bcx-bd.$

12. $\sqrt{10x+11}=44-5x.$

13. $2x+\sqrt{x^2-7}=5.$

14. $x+\sqrt{bx+a^2}=a.$

15. $2\sqrt{\frac{x-3a}{3}}+\sqrt{6x}=\frac{21a+5x}{\sqrt{3x-9}}.$

16. $\sqrt{\frac{3x}{x+1}}-\sqrt{\frac{x+1}{3x}}=2.$

An equation of the form $ax^2+bx-c=0$ may be solved without dividing every term of it by the coefficient of x^2. The side containing the simple quantity may be made a complete square by multiplying the equation by 4 times the coefficient of x^2, and then adding to each side the square of the coefficient of the simple power.

The above equation by this method would become

$$4a^2x^2+4abx+b^2=b^2+4ac,$$

and taking the square root, $2ax+b=\pm\sqrt{b^2+4ac},$

$$\therefore \quad x=\frac{-b\pm\sqrt{b^2+4ac}}{2a}.$$

The advantage of this method is, that no fractions are introduced into the solution, which is frequently the case when the coefficient of x^2 is made equal to $+1$.

The principle of this process may be thus seen :

Making coefficient of x^2 in equation above, $+1$,

$$x^2+\frac{b}{a}x=\frac{c}{a},$$

and solving, we get $x+\frac{b}{2a}=\frac{\sqrt{b^2+4ac}}{2a},$

and multiplying by $2a$, $2ax+b=\sqrt{b^2+4ac}$;

squaring, $4a^2x^2+4abx+b^2=b^2+4ac,$

an equation exactly similar to the one obtained by the above method, from which we obtain the value of x.

EXERCISE LII.

Given : 1. $3x^2+2x=40.$

2. $6x^2+7x=49.$

3. $6x-30=3x^2.$

4. $\frac{2x+3}{10-x}=\frac{2x}{25-3x}.$

5. $8x^2-7x+34=0.$

6. $6x^2=37x-56.$

When a quadratic is reduced to the form $x^2+px+q=0$, the sum of the values of x is equal to the coefficient of x with its sign changed, and the product of the values of x is equal to the third term q.

Let α and β represent the values

$$\alpha=-\tfrac{1}{2}p+\sqrt{\tfrac{1}{4}p^2-q},$$
$$\beta=-\tfrac{1}{2}p-\sqrt{\tfrac{1}{4}p^2-q},$$

adding, $\alpha+\beta=-p$; and multiplying, $\alpha\beta=\frac{p^2}{4}-\left(\frac{p^2}{4}-q\right)=q$.

Also in $ax^2+bx-c=0$, $x^2+\frac{b}{a}x-\frac{c}{a}=0$, $-\frac{b}{a}=$ sum and $-\frac{c}{a}=$ product of values of x.

If α and β be the two roots of the quadratic $x^2+px+q=0$, then $(x-\alpha)(x-\beta)=x^2+px+q$.

$$\alpha=-\frac{p}{2}+\sqrt{\frac{p^2}{4}-q},\quad \beta=-\frac{p}{2}-\sqrt{\frac{p^2}{4}-q},$$

then $x-\alpha=x+\frac{p}{2}-\sqrt{\frac{p^2}{4}-q}$, and $x-\beta=x+\frac{p}{2}+\sqrt{\frac{p^2}{4}-q}$,

and multiplying, $(x-\alpha)(x-\beta)=\left(x+\frac{p}{2}\right)^2-\left(\frac{p^2}{4}-q\right)=$

$$=\left(x^2+px+\frac{p^2}{4}\right)-\left(\frac{p^2}{4}-q\right)=x^2+px+q.$$

If therefore on inspection of a quadratic we see a quantity of the form $x-a$ which will divide the equation without a remainder, we know one root of the equation, and by dividing by this we obtain the other root of the form $x-\beta$. This remark applies to equations of all degrees. If α be a root of the equation, $x-\alpha$ will divide the equation without a remainder. Hence an equation of a cubic form may sometimes be reduced to a quadratic by dividing by $x-\alpha$, that is, by expelling from it one of the roots, and leaving a quadratic to solve.

From $(x-\alpha)(x-\beta)=x^2+px+q$, it is evident that we can form an equation having the roots given.

Thus, with roots 2 and -10, $(x-2)(x+10)=x^2+8x-20=0$; with roots $\frac{2}{5}$ and $\frac{3}{5}$, $(x-\frac{2}{5})(x-\frac{3}{5})=x^2-x+\frac{6}{25}=0$.

EXERCISE LIII.

It is required to resolve the following into their original factors :

1. $x^2+2x-63=0$.
2. $2x^2-5x-228=0$.
3. $5x^2-3x-8=0$.
4. $2(3x^2-\sqrt{6})-(3\sqrt{2}-4\sqrt{3})x=0$.
5. $3x^2+x\sqrt{5}-10=0$.
6. $abx^2-(a^2-b^2)x-ab=0$.
7. $2x^2+3ax+3a-2=0$.
8. $(m-2)x^2+3x-(m-5)=0$.
9. $15m^2-16m-15=0$.
10. $9m^2-49=0$.
11. $a^2+b^2=0$.
12. $abcx^2-a^2b^2x+c^4x-abc^3=0$.

Exercise LIV.

Form equations with roots:

1. 5 and -6.
2. $\frac{1}{3}$ and $-\frac{4}{3}$.
3. 3 and -3.
4. 5 and $-\frac{1}{5}$.
5. 4 and $-\frac{3}{2}$.
6. $2+\sqrt{3}$ and $2-\sqrt{3}$.
7. 3 and $\frac{19}{6}$.
8. 4 and $-\frac{45}{11}$.

Similarly a cubic, biquadratic, or equation of any dimension may be formed, given the roots.

Thus the cubic whose roots are 2, 3, 4 is

$$(x-2)(x-3)(x-4)=x^3-9x^2+26x-24=0\,;$$

with roots -1, 2, 7,

$$(x+1)(x-2)(x-7)=x^3-8x^2+5x+14=0.$$

The biquadratic whose roots are 1, 2, 3, 4 is

$$(x-1)(x-2)(x-3)(x-4)=x^4-10x^3+35x^2-50x+24=0.$$

Form equations with roots:

9. $-3, -5, 8$.
10. $2, 5, 11$.
11. $1, -2, -6, 7$.
12. $\frac{1}{5}, -\frac{3}{10}, -\frac{3}{2}$.
13. $\frac{1}{2}, \frac{1}{4}, 1, 3$.
14. $5, 4+\sqrt{7}, 4-\sqrt{7}$.
15. $3, 3, -3, +\sqrt{-1}, -\sqrt{-1}$.

The equations hitherto treated of have, after the reductions, assumed the form $x^2+px-q=0$. But all equations of the form $x^{2m}+px^m-q=0$, in which the exponent of one of the powers of the unknown quantity is double that of the other, may be solved by the ordinary method of solving a common adfected quadratic.

$$\text{Since} \qquad \left(x^m+\frac{p}{2}\right)^2=x^{2m}+px^m+\frac{p^2}{4},$$

$$\text{we have} \qquad \left(x^m+\frac{p}{2}\right)^2-\frac{p^2}{4}=x^{2m}+px^m.$$

By substituting this in the original equation,

$$x^{2m}+px^m-q=0,$$

we have $\left(x^m+\frac{p}{2}\right)^2-\frac{p^2}{4}-q=0,$

$$\therefore \quad \left(x^m+\frac{p}{2}\right)^2=\frac{p^2}{4}+q,$$

$$\text{or} \qquad x^m+\frac{p}{2}=\pm\sqrt{\frac{p^2}{4}+q},$$

$$\therefore \qquad x=\sqrt[m]{-\frac{p}{2}\pm\sqrt{\frac{p^2}{4}+q}}=\sqrt[m]{\frac{-p\pm\sqrt{p^2+q}}{2}}.$$

Or otherwise. A new unknown quantity may be assumed equal to the lower power in the given equation. The square of this new quantity will be equal to the higher power, and thus a quadratic of the form $x^2+px-q=0$ will be formed.

Ex.
$$x^6-6x^3=16,$$
$$x^6-6x^3+9=25,$$
$$x^3-3=\pm 5,$$
$$\therefore \quad x^3=8 \text{ or } -2,$$
$$\therefore \quad x=2 \text{ or } \sqrt[3]{-2}.$$

Again. Assume $y=x^3$, then $y^2=x^6$,
$$\text{and} \quad y^2-6y=16,$$
$$y^2-6y+9=25,$$
$$y-3=\pm 5, \text{ and } y=8 \text{ or } -2.$$

Having found y, x is easily obtained, for $x^3=y$, $\therefore x=\sqrt[3]{y}$, $\therefore x=\sqrt[3]{8}$ or 2, and $\sqrt[3]{-2}$.

This method of substituting a new unknown quantity greatly simplifies a solution when the unknown quantity in the given equation is a compound one.

Ex. $$2x^2+\sqrt{2x^2+1}=11.$$

Adding 1 to both sides, $2x^2+1+\sqrt{2x^2+1}=12$.

Let $y=\sqrt{2x^2+1}$, then $y^2=(2x^2+1)$, and the equation becomes $y^2+y=12$, and solving for y, $y=3$ or -4. Now in order to find x, we must equate the values of y with $2x^2+1$. Taking the first value of y, we have

$$2x^2+1=9, \text{ whence } x=\pm 2,$$

and taking the second value of y, $2x^2+1=16$, whence $x=\pm\sqrt{\tfrac{15}{2}}$.

EXERCISE LV.

1. $x^4-74x^2+1225=0$.
2. $x^4-8=7x^2$.
3. $x^4-25x^2+144=0$.
4. $3\tfrac{1}{2}x^4+24\tfrac{1}{3}x^2=502\tfrac{1}{2}$.
5. $x^4+b^2c^2=(2bc+4a^2)x^2$.
6. $2\{a-\sqrt{(x^2-a^2)}\}^2=x^2$.
7. $\dfrac{1107}{x^3}-x^3=14$.
8. $x^3+6=\dfrac{1}{x^3}$.
9. $\tfrac{1}{9}x^6-12(x^3-2)=0$.
10. $x^6-90x^3+343=0$.
11. $x^8-2x^4-1=0$.
12. $228+\sqrt{(x^5-18)}=x^5$.
13. $\dfrac{a^2x^2}{\sqrt{(x^2+b^2)}}-a^2\sqrt{(x^2+b^2)}=b^2x$.
14. $x-\sqrt{x}=30$.
15. $2x-\sqrt{x}=45$.
16. $3x+\sqrt{x}=52$.
17. $x^2+1=2+\sqrt{(x^2+5)}$.
18. $(x^2-a^2)^2-2b(x^2-a^2)=c^2$.

19. $(\frac{1}{3}x)^2+1=(\frac{5}{13})^2-\frac{10}{13}x-(\frac{1}{4}x)^2.$

20. $\dfrac{1}{x+\dfrac{1}{x}}=1.$

21. $x^2=2x\sqrt{-1}-1.$

22. $\dfrac{x^2}{(m+n)^2}-\dfrac{4mn}{(m+n)^2}x+(m-n)^2=0.$

23. $(3x-25)(7x+29)=0.$

24. $(x-\sqrt{-7})(x-\sqrt{-11})=0.$

25. $(x-1\frac{2}{3})(x-4\frac{5}{6})+(x-7\frac{8}{9})(x-10\frac{11}{12})=129\frac{581}{675}.$

26. $\dfrac{1}{x+1}-\dfrac{1}{x+2}=\frac{1}{6}.$

27. $\dfrac{x-1}{x-2}-\dfrac{x-3}{x-4}=-\frac{2}{3}.$

28. $\dfrac{2+3x}{1-4x}-\dfrac{6-5x}{7x-25}=\dfrac{16-x}{28x-193}.$

29. $\dfrac{1}{a-x}-\dfrac{1}{a+x}=\dfrac{3+x^2}{a^2-x^2}.$

30. $\dfrac{12x^3-11x^2+10x-78}{8x^2-7x+6}=1\frac{1}{2}x-\frac{1}{2}.$

31. $\dfrac{x^4+12x^3+3x^2+7x+1}{x^2+7x+4}=x^2+5x-4.$

32. $x^2+\dfrac{a-b}{b}=\dfrac{14a^2-5b(a+2b)}{18ab}+\dfrac{2a-3b}{2ab}x.$

33. $\dfrac{x}{a-b}=\dfrac{1}{2\sqrt{a-x}}.$

34. $(13x^2)^2+(12x)^2=5^2.$

35. $\sqrt{x-1}=x-1.$

36. $x+\sqrt{x}=20.$

37. $\sqrt{\frac{1}{2}}=2x\sqrt{1-x^2}.$

38. $x-(a+b)\sqrt{x}=2a(a-b).$

39. $x+ab=(a+b)\sqrt{x}+2(a-b)^2.$

40. $\dfrac{\sqrt{x}}{21-\sqrt{x}}+\dfrac{21-\sqrt{x}}{\sqrt{x}}=2\frac{1}{2}.$

Quadratic Equations with Two Unknown Quantities.

Many of these are very complicated. We will proceed to an examination of the solution of the principal cases, and add a numerous selection of examples for practice. The ingenuity of the student will suggest some method of solution in cases for which no definite rules can be given.

The ordinary method of solution is to substitute in one of the equations the value of x or y obtained from the other equation. But,

1. When the sum a, and the product b, of two quantities are given, to find the quantities.

$$x+y=a,\quad xy=b.$$

Substituting the value of y from the second equation, the first becomes

$$x+\frac{b}{x}=a,$$

$$x^2-ax=-b,$$

$$x^2-ax+\tfrac{1}{4}a^2=\tfrac{1}{4}a^2-b,$$

$$x=\tfrac{1}{2}a\pm\sqrt{(\tfrac{1}{4}a^2-b)},$$

$$y=\tfrac{1}{2}a\mp\sqrt{(\tfrac{1}{4}a^2-b)}.$$

Or thus,

$$\begin{array}{l} x^2+2xy+y^2=a^2 \\ \quad 4xy \qquad\;\; =4b \\ \hline x^2-2xy+y^2=a^2-4b. \end{array}$$

From which and the first of the two equations, we have

$$x+y=a,\ \ x-y=\pm\sqrt{(a^2-4b)},$$

adding, $$x=\frac{a\pm\sqrt{(a^2-4b)}}{2}=\tfrac{1}{2}a\pm\sqrt{\tfrac{1}{4}a^2-b},$$

subtracting, $$y=\frac{a\mp\sqrt{(a^2-4b)}}{2}=\tfrac{1}{2}a\mp\sqrt{\tfrac{1}{4}a^2-b}.$$

2. From the sum of two quantities a, and the sum of their squares b, to find the quantities.

$$x+y=a,\ \ x^2+y^2=b.$$

Determining the value of y in the first equation, and substituting it in the second, we have

$$x^2+(a-x)^2=b,$$
$$\therefore\ \ 2x^2-2ax=b-a^2,$$
$$x^2-ax=\frac{b-a^2}{2},$$
$$x^2-ax+\tfrac{1}{4}a^2=\frac{2b-a^2}{4},$$
$$x=\tfrac{1}{2}a\pm\tfrac{1}{2}\sqrt{2b-a^2},$$
$$y=\tfrac{1}{2}a\mp\tfrac{1}{2}\sqrt{2b-a^2}.$$

This example is easily reduced to the form of the foregoing one, and may be solved in a similar manner.

Thus, $x^2+2xy+y^2=a^2$, and $2xy+b=a^2$,

$$\therefore\ \ xy=\frac{a^2-b}{2},\ \text{and}\ x+y=a.$$

Therefore $x+y=a$, and $xy=\frac{a^2-b}{2}$ give us the sum of two quantities and their product.

3. When the difference a, and the sum of the squares of two quantities are given, to find the quantities.

$$x-y=a,\ \ x^2+y^2=b.$$

Proceeding as in the last example, we get

$$x=\tfrac{1}{2}a\pm\tfrac{1}{2}\sqrt{2b-a^2},\ \ y=-\tfrac{1}{2}a\pm\tfrac{1}{2}\sqrt{2b-a^2}.$$

4. When the product a, and the sum of the squares of two quantities are given, to find the quantities.

$$xy=a,\ \ x^2+y^2=b,$$
$$x^2+2xy+y^2=b+2a,$$
$$x^2-2xy+y^2=b-2a,$$
$$x+y=\pm\sqrt{b+2a},$$
$$x-y=\pm\sqrt{b-2a}.$$

Adding, $$2x=\pm\sqrt{b+2a}\pm\sqrt{b-2a},$$

and subtracting, $$2y=\pm\sqrt{b+2a}\mp\sqrt{b-2a},$$

from which, by dividing by 2, the values of x and y are determined.

5. An equation of the form $x+y=a$, $x^2+y^2+xy=b$, is thus solved:

$$\begin{array}{ll} & x^2+y^2+2xy=a^2,\\ & \underline{x^2+y^2+xy=b,}\\ \text{subtracting,} & xy=a^2-b,\end{array}$$

so that we have an equation similar to that solved in case 1, $x+y=a$, $xy=a^2-b$.

6. To solve equations of the form $x+y+x^2+y^2=a$, $x-y+x^2-y^2=b$.

Adding and subtracting, we obtain

$$2x^2+2x=a+b,$$
$$2y^2+2y=a-b,$$

quadratics involving *one* unknown quantity, which may be easily solved.

7. When the sum a of two quantities, and the sum b of their cubes are given, to find the quantities.

$$x+y=a,\quad x^3+y^3=b.$$

Here, dividing the second equation by the first, and squaring the first, we obtain

$$\begin{array}{ll} & x^2-xy+y^2=\frac{b}{a},\\ & \underline{x^2+2xy+y^2=a^2,}\\ \text{subtracting,} & 3xy \quad =a^2-\frac{b}{a}.\end{array}$$

But by the first equation $x+y=a$; therefore $x+y=a$, and $3xy=a^2-\frac{b}{a}$ give us an example whose method of solution has been shown in case 1.

8. Given the sum of two quantities multiplied by the sum of their squares $=a$, and their difference multiplied by the difference of their squares $=b$, to find the quantities.

$$(x+y)(x^2+y^2)=a,$$
$$(x-y)(x^2-y^2)=b.$$

Let $x+y=s$, and $xy=p$,

then $x^2+y^2+2xy=s^2$, $x^2+y^2=s^2-2p$,

$$(x-y)^2=x^2+y^2-2xy=s^2-4p.$$

Also because $(x-y)(x^2-y^2)=(x-y)^2(x+y)$, it follows from the given equations:

$$(a)\quad s(s^2-2p)=a,\quad s(s^2-4p)=b,$$
$$s^2-2p=\frac{a}{s},\qquad s^2-4p=\frac{b}{s},$$
$$2s^2-4p=\frac{2a}{s},\qquad s^2-4p=\frac{b}{s};$$

subtracting these two equations we obtain $s^2=\frac{2a-b}{s}$, and $s^3=2a-b$, therefore $s=\sqrt[3]{2a-b}$. But $x+y=s$, $\therefore$ $x+y=\sqrt[3]{2a-b}$, and (a) $p=\frac{s^3-a}{2s}$. So that now we have the two equations $x+y=s$, and $xy=p$, from which, since s and p are known quantities, x and y may be found by case 1.

Exercise LVI.

1. $\left.\begin{array}{l} x^2+y^2=13, \\ x^2-y^2=-10{\cdot}12. \end{array}\right\}$

2. $\left.\begin{array}{l} (13x)^2+2y^2=177, \\ (2y)^2-13x^2=3. \end{array}\right\}$

3. $\left.\begin{array}{l} xy=a, \\ \dfrac{x}{y}=b. \end{array}\right\}$

4. $\left.\begin{array}{l} \dfrac{x+y}{x-y}=\frac{193}{11}, \\ \dfrac{19}{\frac{1}{9}x}=\dfrac{\frac{1}{8}y}{41}. \end{array}\right\}$

5. $\left.\begin{array}{l} (x^2+y^2):(x^2-y^2)=25:7, \\ xy=48. \end{array}\right\}$

6. $\left.\begin{array}{l} 14x^2-122y^2=120, \\ x=3y. \end{array}\right\}$

7. $\left.\begin{array}{l} x^2+xy=a, \\ yx+y^2=b. \end{array}\right\}$

8. $\left.\begin{array}{l} \dfrac{a}{x^2}-by^2=(a-b)^3, \\ \dfrac{b}{x^2}+ay^2=(a^2-b^2)(a-b). \end{array}\right\}$

9. $\left.\begin{array}{l} 2(x+4)^2-5(y-7)^2=75, \\ 7(x+4)^2+15(y-7)^2=1075.^* \end{array}\right\}$

10. $\left.\begin{array}{l} \left(\dfrac{9}{x}\right)^2=\left(\dfrac{25}{y}\right)^2-16, \\ \dfrac{9}{x}=\dfrac{25}{y}. \end{array}\right\}$

11. $\left.\begin{array}{l} \left(\dfrac{24}{x}\right)^2+(y-4)^2=65, \\ \left(\dfrac{12}{x}\right)^2+9=(5y-20)^2. \end{array}\right\}$

12. $\left.\begin{array}{l} (x-2)(y-3)=1, \\ \dfrac{x-2}{y-3}=1. \end{array}\right\}$

13. $\left.\begin{array}{l} (x+y)^2-2x^2=49, \\ 3x^2+4(x+y)^2=372. \end{array}\right\}$

14. $\left.\begin{array}{l} x^2+y\sqrt{xy}=336, \\ y^2+x\sqrt{xy}=112. \end{array}\right\}$

15. $\left.\begin{array}{l} x+y=s, \\ xy=p. \end{array}\right\}$

16. $\left.\begin{array}{l} x-y=2d, \\ xy=p. \end{array}\right\}$

17. $\left.\begin{array}{l} \dfrac{1}{x}+\dfrac{1}{y}=\dfrac{1}{5}, \\ \dfrac{10}{xy}=\dfrac{1}{18}. \end{array}\right\}$

18. $\left.\begin{array}{l} (7+x)(6+y)=80,^\dagger \\ x+y=5. \end{array}\right\}$

19. $\left.\begin{array}{l} (x^2+2y^2)(3x^2-4y^2)=48,^\S \\ 2x^2-y^2=7. \end{array}\right\}$

20. $\left.\begin{array}{l} 6:x=y:10, \\ x-y=11. \end{array}\right\}$

21. $\left.\begin{array}{l} x+y=40, \\ x+y^2=546. \end{array}\right\}$

22. $\left.\begin{array}{l} (x-4)^2+(y+4)^2=100, \\ x+y=14. \end{array}\right\}$

23. $\left.\begin{array}{l} x+y=134154, \\ \sqrt{x}+\sqrt{y}=468. \end{array}\right\}$

24. $\left.\begin{array}{l} 4:x=y:42, \\ \sqrt[4]{x^2+y^2}=5. \end{array}\right\}$

25. $\left.\begin{array}{l} \dfrac{1}{xy}=12, \\ \dfrac{x^2+y^2}{x^2y^2}=25. \end{array}\right\}$

26. $\left.\begin{array}{l} 12:x=y:3, \\ \sqrt{x}+\sqrt{y}=5. \end{array}\right\}$

27. $\left.\begin{array}{l} x^2-y^2+x+y=0{\cdot}375, \\ x^2-y^2-(x-y)={\cdot}125. \end{array}\right\}$

28. $\left.\begin{array}{l} (3x+4y)(7x-2y)+3x+4y=44, \\ (3x+4y)(7x-2y)-7x+2y=30. \end{array}\right\}$

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* Let $x+4=z$, $y-7=u$. Determine the values of z and u, and from these x and y.

† Put $7+x=z$, and $6+y=u$.

§ Put $x^2+2y^2=z$, $3x^2-4y^2=u$.

29. $\left.\begin{array}{l} xy=\frac{1}{a^2-b^2}, \\ x^2+y^2+xy=\frac{3a^2+b^2}{(a^2-b^2)^2}. \end{array}\right\}$

30. $\left.\begin{array}{l} x^2-y^2=m, \\ y(x+y)=n. \end{array}\right\}$

31. $\left.\begin{array}{l} -x^2+6xy-9y^2+4x-12y=4, \\ x^2-2xy+3y^2-4x+5y=53. \end{array}\right\}$

32. $\left.\begin{array}{l} x+y=a, \\ x^3+y^3=b. \end{array}\right\}$

33. $\left.\begin{array}{l} \sqrt[3]{x}+\sqrt[3]{y}=12, \\ x+y=468. \end{array}\right\}$

34. $\left.\begin{array}{l} x-y=a, \\ x^3-y^3=b. \end{array}\right\}$

35. $\left.\begin{array}{l} x^2-y^2=14896, \\ \sqrt[3]{x^2}-\sqrt[3]{y^2}=\sqrt[3]{64^2}. \end{array}\right\}$

36. $\left.\begin{array}{l} \sqrt{x}+\sqrt{y}=12, \\ x^2+y^2=4026. \end{array}\right\}$

37. $\left.\begin{array}{l} x^2y-xy^2=30, \\ x^3y^2-x^2y^3=450. \end{array}\right\}$

38. $\left.\begin{array}{l} 2(x^2+y^2)(x+y)=15xy, \\ 4(x^4-y^4)(x^2-y^2)=45x^2y^2. \end{array}\right\}$

39. $\left.\begin{array}{l} ax+by=2(x^2-y^2), \\ \frac{b}{x-y}-\frac{a}{x+y}=\frac{x^2+y^2}{xy}. \end{array}\right\}$

40. $\left.\begin{array}{l} \frac{(2x-1)(2y-1)+1}{x^2-y^2+2y-1}=a+b, \\ \frac{y^2-(x-1)^2}{x^2-(y-1)^2}=ab. \end{array}\right\}$

Problems producing Quadratic Equations.

1. Find two numbers, one of which is triple of the other, and the sum of whose squares is 90.

Let $x=$ one number, then $3x=$ the other,

$$\text{and}\quad x^2+9x^2=90,\ \text{or}\ 10x^2=90,$$

$$\therefore\quad x^2=9,\ \text{and}\ x=\pm 3,\ \text{one number, and}\ \pm 9=\ \text{the other.}$$

2. A man bought a certain number of ells of cloth for 240 francs. If with the same sum he had bought 3 ells less, it would have cost him 4 francs an ell more. How many ells did he buy?

Let $x=$ number of ells, then $\frac{240}{x}=$ the price of one ell,

$$\text{and}\ \frac{240}{x-3}=\frac{240}{x}+4,\ \text{and solving,}\ x=15.$$

3. Divide the number 20 into two such parts, that the product of the whole and one of the parts may be equal to the square of the other part.

Let $2x=$ the difference of the two parts. Now, 20 being the sum,

$$10+x=\ \text{the greater part,}$$
$$10-x=\ \text{the lesser part.}$$

By equation, $20(10-x)=(10+x)^2$, and solving, we obtain

$$\text{greater part,}\quad 10+x=10\sqrt{5}-10=10(\sqrt{5}-1),$$
$$\text{lesser part,}\quad 10-x=30-10\sqrt{5}=10(3-\sqrt{5}).$$

From this we see that a number cannot be divided into two *rational* parts, so as to answer the conditions of the problem.

4. Divide the quantity m into two such parts that their product may be n^2.

Let $x=$ one part, then $m-x=$ the other,

$$\text{and } x(m-x)=n^2\text{; solving, we shall obtain } x=\frac{m\pm\sqrt{m^2-4n^2}}{2},$$

$$\therefore\ m-x=\frac{m\mp\sqrt{m^2-4n^2}}{2}.$$

Here we see that if $4n^2$ is $>m^2$, the two parts are impossible, and therefore the problem is impossible when the proposed product is $>$ the square of half the given quantity.

EXERCISE LVII.

1. Find a number the fifth part of which multiplied into the seventh equals 4235.

2. If I multiply the number of shillings I have in my pocket by itself I shall get $132\frac{1}{4}$. How many do I possess?

3. I multiply $3\frac{4}{7}$ times a certain number by 8·68 times the same number, and obtain 5239. Required the number.

4. Find two numbers which are in the proportion of 11 : 13, and whose product is 7007.

5. There is a number, the product of whose third, fourth, and fifth parts is equal to its sixth part. Required the number.

6. Find three numbers in the proportion of $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, the sum of whose squares is equal to 10309.

7. There is a rectangular field whose length is 3367, and breadth 37 yards, and another of the same area whose length is to its breadth as 7 : 13. Find the length and breadth of the latter.

8. Resolve the quantity a into two factors in the proportion of p to q.

9. If 3 be added to a number, and 3 also subtracted from it, then the sum of the quotients obtained on dividing the larger number by the smaller, and the smaller number by the larger, is equal to $3\frac{1}{3}$. What is the number?

10. Divide the number 20 into two parts, such that their squares may be as $1 : 2\frac{1}{4}$.

11. What is the side of a square, the area of which would be $\frac{9}{16}$ times larger, were the side lengthened 3 feet?

12. A square vineyard, in which the shoots are set at right angles, so that each pair is 4 feet apart, is to be so altered in arrangement that the shoots may stand only $3\frac{1}{2}$ feet apart. If to effect this 8640 shoots are necessary, how many feet long is each side of the vineyard?

13. In a right-angled triangle the hypothenuse is 1000 feet long; required the sides, one of which is $3\frac{3}{7}$ times longer than the other.

14. If I add to the first factor of the product 6·52 a certain number, and from the second factor subtract the same number, I shall obtain as the product of the two new factors 35 times the number which was added to the first factor. Required the number.

15. What number added to its reciprocal gives 2·9 ?

16. What number subtracted from its reciprocal gives 6·09 ?

17. A looking-glass 33 inches high, and 22 broad, is to have a frame of uniform width, but of such dimensions that the areas of the frame and glass are equal. What must be the width of the frame in inches ?

18. Two bodies start from two points, A and B (whose distance is 910 feet), at a uniform pace, towards each other. If the first starts 56 seconds before the other, they will meet midway; but if both bodies start at the same time from A and B, after 20 seconds the distance between them will be 550 feet. In how many seconds would each body pass from A to B ?

19. There are two square pieces of ground to be planted with trees also in the form of squares. If on the first piece the trees are planted 5 feet apart, and on the second $4\frac{1}{2}$ feet, 11113 will be required. But if on the first the trees are planted $5\frac{1}{2}$ feet apart, and on the second 6 feet apart, 7816 will be wanted. How many feet long is each piece of ground ?

20. The area of the floor of a room is 273 feet, that of one of the sides 189, and that of the other 117. What are the length, breadth, and height of the room ?

21. The contents of two cubic vessels are 407 inches, the height of the two together is 11 inches. What are the contents of each vessel ?

22. If I add 2 to the numerator of a certain fraction, and subtract 2 from the denominator, I obtain the reciprocal value of the fraction. But if I subtract 2 from the numerator, and add 2 to the denominator, I shall obtain a number which, increased by $1\frac{1}{15}$, will be equal to the reciprocal value of the fraction sought. What is the fraction ?

23. A piece of cloth shrinks, on being damped with water, $\frac{1}{8}$ in length and $\frac{1}{16}$ in breadth. If a piece of cloth shrinks in area $5\frac{3}{4}$ square yards, and in circumference $4\frac{1}{4}$ yards, what are its length and breadth ?

24. A besieged town can hold out but 10 days longer; if 120 men depart, and each receives daily $\frac{5}{8}$ of a pound of bread less, the town can hold out 16 days; it will be able to hold out for the same time if 200 men depart, and each receive daily $\frac{3}{8}$ of a pound less. Of how many men is the garrison composed, and how much bread does each man receive daily ?

25. A certain number of labourers can remove a heap of stones in 8 hours from one place to another: if there were 8 more labourers, and each carried every time 5 lbs. less, the heap would be removed in 7 hours: but if the labourers were 8 fewer in number, and each carried every time 11 lbs. more, the heap would be removed in 9 hours. How many men are employed, and how much does each carry ?

26. There are two numbers whose sum is 63. If the quotient arising from dividing the greater by the less be multiplied by the greater, and to the product $20\frac{1}{4}$ be added, there will result a number the cube root of which will be 1 less than the seventh part of the greater number. Required the two numbers.

27. A cistern can be filled from four pipes in $115\frac{1}{5}$ minutes. It is found that to fill it from the second alone requires 4 hours longer than to

fill it from the first, 8 hours longer from the third, and 12 hours longer from the fourth. In what time can it be filled from the first?

28. A person bought coffee for 88*s*., and sugar for a similar sum, and obtained of the latter 90 lbs. more than of the former. He sold 29 lbs. of coffee, and 57 lbs. of sugar, and gained (at the rate of 20 per cent) on the whole 31*s*. How many lbs. of coffee and of sugar did he buy?

29. What are the contents of the greatest square whose circumference is n feet?

30. Four persons, A, B, C, D, have each a certain number of pounds. B has one more than A, C one more than B, and D one more than C. If we multiply the four sums together we shall obtain a product 1168*l*. more than D's pounds cubed. Required the sum possessed by each person.

CHAPTER XIII.

INDETERMINATE EQUATIONS.

A PROBLEM is said to be *indeterminate* or *unlimited* when the equations expressing its conditions are fewer in number than the unknown quantities. Strictly speaking, such kinds of problems are capable of innumerable solutions, but the *integral* solutions to which the question is commonly restricted are limited to a determinate number. To assist in the discovery of these solutions, suppose $\frac{ax \pm b}{c}$ an algebraic fraction in its lowest terms, x being indeterminate, and a, b, c any given integers; then the least integral value of x that will give $\frac{ax \pm b}{c}$ also an integer, will be found by the following rules.

Divide the denominator c by a, the coefficient of the indeterminate quantity, and that divisor by the remainder, and so on, until a unit only remains. Write down the successive quotients in a line, under the first of which place a unit, and under the second write the first; under the third place the product of the two second terms, plus the first of the lower line; under the fourth put the product of the two third terms, plus the second term of the lower line; and so on, until every term in the upper line has been multiplied.

Then multiply the last number thus found by the quantity b, and divide the product by c; the remainder will be the value of x required, provided the number of terms in the upper line be even, and the sign of $b-$, or the number odd, and the sign of $b+$; but if the number of terms be even, and the sign of $b+$, or odd, and the sign of $b-$, then the difference between the remainder and c will be the least integral value of x.

a is supposed to be less than c, and these two numbers prime to each other; for if they have a common factor by which b is not divisible, no integer could be assigned to x, so as to give $\frac{ax \pm b}{c}$ an integer, the reason of which, and the rules preceding, will presently be seen.

Examples to illustrate the above.

1. Let the fraction be $\frac{87x-50}{256}$, then the quotients set down in order will be

2, 1, 16, 2,
1, 2, 3, 50, 103, and $\frac{103\times 50}{256}=20$, with remainder 30,

which is the least integral value of x.

2. $\frac{71x+10}{89}$. The quotients are

1, 3, 1,
1, 1, 4, 5, and $5\times 10=50=x$.

3. $\frac{377x-250}{450}$. Here the quotients are

1, 5, 6,
1, 1, 6, 37, and $\frac{37\times 250}{450}=20$, with remainder $200=x$.

4. To find the least number which, when divided by 17 and 26, shall leave remainders 7 and 13 respectively.

Let N, the number sought, $=17x+7$,

then $\frac{17x+7-13}{26}=\frac{17x-6}{26}$ must be integral, and the least integral value of x by the rule $=8$, and $\therefore 17x+7=143=N$.

5. Find all the positive integral solutions of x and y in the equation $9x+13y=2000$.

Transposing and dividing by coefficient of x, we have

$$x=\frac{2000-13y}{9}=222-y+\frac{2-4y}{9},$$

in which, since x is integral, $\frac{2-4y}{9}$ is also integral, from which the least integral value of $y=5$, and $\therefore x=215$.

The remaining solutions will be found by adding 9 continually to the last value of y, and subtracting 13 from that of x. The solutions will be 17 in number, as shown by the following table:

$x=215$	202	189	176	163	150	137	124	111	98	85	72	59	46	33	20	7
$y=$ 5	14	23	32	41	50	59	68	77	86	95	104	113	122	131	140	149

Similarly, the least value of y and the greatest of x being found, the rest of the values will be obtained by adding the coefficient of x in the given equation to the least value of y, and subtracting the coefficient of y from the corresponding value of x. *Hence, if the greatest value of* x *be divided by the coefficient of* y, *the remainder will be the least value of* x, *and the quotient* $+1$ *will give the number of solutions.*

6. Let $87x+256y=15410$. Find the least positive integral value of x, and the greatest of y.

By transposition and division, as before, $y=\frac{15410-87x}{256}=60-\frac{87x-50}{256}$, from which, as in example 1, $x=30$, whence y may likewise be determined. We will determine these values, however, by a method on which the demonstration of the rules before given depends.

It is evident, since the quantity $87x-t$ $(t=50)$ is divisible by 256, that its double $174x-2t$ will also be divisible by the same number. But $256x$ is divisible by 256, and if from $256x$ we subtract $174x-2t$, the remainder $82x+2t$ will also be divisible by 256 (whatever number measures the whole and one part of another number must measure also the remaining part); therefore if the quantity last found be subtracted from the first, the remainder $5x-3t$ will be divisible by 256. And if, in order to diminish further the coefficient of x, this remainder be multiplied by 16, and subtracted from the preceding one, the difference $2x+50t$ will still be divisible by 256. In like manner, the double of $2x+50t$ being subtracted from the preceding, we obtain $x-103t$, a quantity still divisible by 256. But $\frac{103t}{256}=20+\frac{30}{256}$, ∴ $x-30$ must be divisible by 256, and ∴ $x=30$, or some multiple of $256+30$; but 30 being the least value, is the one required.

We will illustrate this process by another example.

Let the quantity be $\frac{987x+651}{1235}$, then make $t=651$; it will stand as follows:

$$\begin{array}{lr}
\text{From} & 1235x \\
\text{take} & 987x+t \\ \hline
\text{1st rem}^{\text{r}} & 248x-t \\
\quad,,\ \times 3= & 744x-3t \\ \hline
\text{2d rem}^{\text{r}} & 243x+4t \\ \hline
\text{3d rem}^{\text{r}} & 5x-5t \\
\quad,,\ \times 48= & 240x-240t \\ \hline
\text{4th rem}^{\text{r}} & 3x+244t \\ \hline
\text{5th rem}^{\text{r}} & 2x-249t \\ \hline
\text{6th rem}^{\text{r}} & x+493t
\end{array}$$

x being without coefficient, let $493t=320943$ be divided by 1235, and the remainder will be 1078, ∴ $x+1078$ is also divisible by 1235, and ∴ the least value of x is 157. The method of working may be varied from the above, it being immaterial whether the last remainder or a multiple of it be taken from the preceding one, or the preceding one from some greater multiple of the last.

To determine whether it is possible to pay 100*l.* in guineas and moidores only.

Let $x=$ guineas, and $y=$ moidores, then

$$21x+27y=2000,\ \therefore\ x=\frac{2000-27y}{21}=95-y-\frac{6y-5}{21}.$$

This fraction being in its lowest terms, and the numbers 6 and 21 having a common measure 3, an integral solution is impossible. The reason of which depends on these considerations: whatever number is divisible by a given number, is divisible also by all the factors of that number; and any quantity which exactly measures the whole and one part of another, measures also the remaining part. $6y-5$ to have the result a whole number should be divisible by 21, and $\therefore$ by 3 (a common measure of a and c), but we see that it is not, and therefore the thing proposed is impossible.

EXERCISE LVIII.

1. Find in how many different ways it is possible to pay 100*l.* in guineas and 17*s.* pieces.

2. A certain number of sheep and oxen were bought for 100*l.*; sheep at 17*s.*, and oxen at 7*l.* each. Required the number of each.

3. The entertainment of a certain number of men and women at an inn cost 33*s.*; each man paid 3*s.* 6*d.*, and each woman 1*s.* 4*d.* Of how many persons of both sexes did the company consist?

4. Find the least integer which, being divided by 28, 19, and 15, shall leave remainders 19, 15, and 11 respectively.

5. Find all the positive integral values of x, y, z in the equation $5x+7y+11z=224$.

6. Find all the positive integral values of x, y, z in the equation $17x+19y+21z=400$.

7. A person walks over a piece of ground measuring between 120 and 130 feet long; he finds by taking strides 3 feet long, there is 1 foot left, and by striding 4 feet each time, there are 3 feet left. Required the length of the ground.

8. I owe a friend 2 guineas and a half, but have only crowns to pay with; my friend has nothing but threepenny pieces; how are we most easily to settle the account?

CHAPTER XIV.

ARITHMETICAL, GEOMETRICAL, AND HARMONICAL PROGRESSION.

A SERIES of quantities proceeding by a common difference is said to be in arithmetical progression. Each quantity is called a *term* of the series.

The general form of an arithmetical progression is

$$a,\ a+d,\ a+2d,\ a+3d,\ a+4d\ \ldots.$$

a the first term, d the common difference, they being either positive or negative.

1. To find the n^{th} term of an arithmetical series.

Let u be the n^{th} term, then

Indices.	Terms.
1	$a=a+(1-1)d,$
2	$a+d=a+(2-1)d,$
3	$a+2d=a+(3-1)d,$
4	$a+3d=a+(4-1)d.$

From which it is evident that $u=a+(n-1)d.$

2. To find the sum s of the n first terms of an arithmetical series. Let a be the first term, d the common difference, then the n^{th} term will be $a+(n-1)d$, and

$$s=a+(a+d)+(a+2d)+\;\ldots\;(u-2d)+(u-d)+u;$$

writing these terms in the reverse order,

$$s=u+(u-d)+(u-2d)+\;\ldots\;(a+2d)+(a+d)+a,$$

adding, $2s=(a+u)n,$

$$s=\frac{(a+u)n}{2}=\{2a+(n-1)d\}\frac{n}{2}.$$

We have now the two equations $u=a+(n-1)d$, and $s=\{2a+(n-1)d\}\frac{n}{2}$, by which, if three of the five magnitudes a, d, n, u, s are given, the remaining two may easily be determined.

Ex. 1. Find the 10th term and the sum of 10 terms of the series 1, 4, 7, 10, &c.

$a=1$, $d=3$, $n=10$, then $u=1+\{(10-1)\times 3\}=1+(9\times 3)=28$, and $s=(1+28)\times \frac{10}{2}=145.$

Ex. 2. Find the fifth term and the sum of five terms of the series 6, $4\frac{1}{2}$, 3, &c.

Here $a=6$, $d=-\frac{3}{2}$, $n=5$,

$$\therefore\quad u=6+(5-1)\times -\tfrac{3}{2}=0, \text{ and } s=\tfrac{5}{2}\{12+(5-1)\times -\tfrac{3}{2}\}=15,$$

u being, as we have seen, $=a+(n-1)d$, and $s=\frac{(a+u)n}{2}=\{2a+(n-1)d\}\frac{n}{2}.$

Ex. 3. There is an arithmetical series the third term of which is 8, and the eighth term 23; find the series.

The formula for the n^{th} term is $a+(n-1)d$, therefore we have

$$a+2d=8\;\ldots\;(1)$$
$$a+7d=23\;\ldots\;(2)$$

subtracting (1) from (2), $5d=15,$

$$\therefore\quad d=3, \text{ and } a=8-6=2.$$

The series therefore is 2, 5, 8, &c.

Ex. 4. Insert four arithmetical means between 17 and 32; that is, form an arithmetical series of 4+2 terms, of which the first shall be 17, and the last 32.

The problem will be answered if we find d the common difference.

$$u=a+(n-1)d,$$
$$\therefore\ 32=17+5d,$$
$$\therefore\ 5d=15, \text{ and } d=3.$$

The means required are 20, 23, 26, 29.

Ex. 5. There is an arithmetical series whose 11th term is four times its 3d term, and the 5th exceeds the 3d by 3.

$$a= \text{ the first term, } d= \text{ difference,}$$
$$a+10d=4(a+2d),\ \therefore\ 3a=2d,$$
$$a+4d=a+2d+3,$$
$$\therefore\ 2d=3,\ d=1\tfrac{1}{2};\ \text{and } 3a=3,\ \therefore\ a=1.$$

The series, then, is 1, $2\frac{1}{2}$, 4, $5\frac{1}{2}$, &c.

Ex. 6. Find 4 numbers in arithmetical progression such that their sum shall be 26, and the sum of their squares 214.

Let $x=$ the second, and y their common difference: then the four numbers will be represented by $x-y$, x, $x+y$, $x+2y$.

Now by the terms of the question

$$(x-y)+x+(x+y)+(x+2y)=26 \ldots\ldots (1)$$
$$(x-y)^2+x^2+(x+y)^2+(x+2y)^2=214 \ldots\ldots (2)$$
$$\text{or}\quad 4x^2+4xy+6y^2=214;$$
$$\text{From (1), } 2x+y=13,$$
$$\therefore\ 4x^2+4xy+y^2=169,$$
$$(2)\quad 4x^2+4xy+6y^2=214,$$
$$5y^2=45,$$
$$\therefore\ y=3, \text{ and from (1) } x=5.$$

$5-3=2$, the first term; and the four numbers are 2, 5, 8, 11.

Ex. 7. Required the number of all the strokes of a clock in 12 hours.

Here $a=1$, $d=1$, $u=12$, to find s.

By the formula, $\frac{12+1}{2}+\frac{(12+1)(12-1)}{2}=s$, *i.e.* $6\frac{1}{2}+71\frac{1}{2}=78$, the number of strokes.

EXERCISE LIX.

Find the last term and the sum of the terms of the following, given first term, common difference, and number of terms :

1. $a=2,\quad d=3,\quad n=17.$ 2. $a=7,\quad d=\frac{1}{4},\quad n=16.$
3. $a=2\frac{1}{2},\quad d=\frac{1}{3},\quad n=100.$ 4. $a=\frac{3}{4},\quad d=\frac{1}{8},\quad n=26.$
5. $a=\frac{5}{7},\quad d=1\frac{2}{3},\quad n=13.$ 6. $a=-7,\quad d=3,\quad n=8.$
7. $a=-6,\quad d=\frac{3}{4},\quad n=30.$ 8. $a=\frac{1}{2},\quad d=-\frac{1}{8},\quad n=20.$
9. $a=-10,\quad d=-2,\quad n=6.$ 10. $a=-\frac{3}{4},\quad d=-\frac{7}{8},\quad n=25.$

Let the following solutions be verified :

Given.	Required.	
$a, d, n,$	u	$u=a+(n-1)d.$
$a, d, s,$		$u=-\frac{1}{2}d\pm\sqrt{\{(a-\frac{1}{2}d)^2+2sd\}}.$
$a, n, s,$		$u=\frac{2s}{n}-a.$
$d, n, s,$		$u=\frac{s}{n}+\frac{(n-1)d}{2}.$
$a, d, n,$	s	$s=\frac{1}{2}n\{2a+(n-1)d\}.$
$a, d, u,$		$s=\frac{u+a}{2}+\frac{(u+a)(u-a)}{2d}.$
$a, n, u,$		$s=\frac{1}{2}n(a+u).$
$d, n, u,$		$s=\frac{1}{2}n\{2u-(u-1)d\}.$
$a, n, u,$	d	$d=\frac{u-a}{n-1}.$
$a, n, s,$		$d=\frac{2(s-an)}{n(n-1)}.$
$a, u, s,$		$d=\frac{(u+a)(u-a)}{2s-u-a}.$
$n, u, s,$		$d=\frac{2(nu-s)}{n(n-u)}.$
$a, d, u,$	n	$n=1+\frac{u-a}{d}.$
$a, d, s,$		$n=\frac{d-2a}{2d}\pm\sqrt{\left\{\frac{2s}{d}+\left(\frac{2a-d}{2d}\right)^2\right\}}.$
$a, u, s,$		$n=\frac{2s}{a+u}.$
$d, u, s,$		$n=\frac{2u+d}{2d}\pm\sqrt{\left\{\left(\frac{2u+d}{2d}\right)^2-\frac{2s}{d}\right\}}.$
$d, n, u,$	a	$a=u-(n-1)d.$
$d, n, s,$		$a=\frac{s}{n}-\frac{(n-1)d}{2}.$
$d, u, s,$		$a=\frac{1}{2}d\pm\sqrt{\{(u+\frac{1}{2}d)^2-2sd\}}.$
$n, u, s,$		$a=\frac{2s}{n}-u.$

EXERCISE LX.

1. The sum of an arithmetical series is 232, the common difference 3, and number of terms 25 : find the 1st term.

2. The sum of an arithmetical series is 280, the common difference $\frac{1}{2}$, and the first term 1 : required the number of terms.

3. How many strokes do the clocks of Italy, which go on to 24, make in one revolution of the index?

4. What is the 20th term, and the sum of the 20 terms of the series whose first term is 7 and common difference -2?

5. What is the 20th term of the series 4, -3, -10, &c., and the sum of the 20 terms?

6. Insert 6 arithmetical means between 2 and -26.

7. Insert 7 arithmetical means between $\frac{1}{2}$ and -1.

8. Insert 4 arithmetical means between 17 and 52.

9. Find four numbers in arithmetical progression, such that their sum shall be 19, and the sum of their squares $121\frac{1}{2}$.

10. There are four numbers in arithmetical progression; the sum of the squares of the extremes is 272, and the sum of the squares of the means is 208. What are the numbers?

11. The base of a right-angled triangle is 12, and its sides are in arithmetical progression : find the other sides.

12. The product of five numbers in arithmetical progression is 10395, and their sum is 35 : what are the numbers?

13. A debt can be discharged in a year by paying 4*s.* the first week, 8*s.* the second, 12*s.* the third, and so on : required the amount of the debt, and the last payment.

14. One hundred stones are placed on the ground at a distance of 2 feet from each other, the first 20 yards from a basket. How far does a person travel who starts from the basket and brings them one by one to it?

GEOMETRICAL PROGRESSION.

A series of quantities is said to be in geometrical progression when each term of it will divide the next succeeding one; the quotients arising from each division being the same.

These quotients are called the *common factors* or *ratios* of the series.

The general form of a geometrical series is

$$a,\ ar,\ ar^2,\ ar^3,\ ar^4,\ ar^5 \ldots$$

a being the first term, and r the common ratio.

1. To find the general or n^{th} term of a geometrical series.

Indices.	Terms.
1 . .	$a=ar^{1-1}$,
2 . .	$ar=ar^{2-1}$,
3 . .	$ar^2=ar^{3-1}$,
4 . .	$ar^3=ar^{4-1}$,
5 . .	$ar^4=ar^{5-1}$,

$$\therefore\ u=ar^{n-1}.$$

2. To find the sum of the n first terms of a geometrical series.

$$s=a+ar+ar^2+\ \ldots\ldots\ ar^{n-2}+ar^{n-1},$$
$$rs=\quad ar+ar^2+ar^3+\ \ldots\ldots\ ar^{n-1}+ar^n.$$

Subtracting the former of these equations from the latter,

$$rs-s=ar^n-a,\quad (r-1)s=a(r^n-1),$$
$$\therefore\quad s=\frac{a(r^n-1)}{r-1},$$

which is the expression for the sum required.

These two equations, $u=ar^{n-1}$, and $s=\frac{ar^n-a}{r-1}=a\frac{r^n-1}{r-1}=\frac{ru-a}{r-1}$, are the fundamental equations of the whole theory of geometrical progression.

Ex. 1. Find the eighth term, and the sum of the eight terms of the geometrical series 1, 2, 4, &c.

$$a=1,\ r=2,\ n=8.$$

Then $u=1\times2^7=128$, the eighth term; $s=\frac{2^8-1}{2-1}=255$, the sum.

Ex. 2. Find the n^{th} term of (x^2-y^2), $(x+y)$, $\frac{x+y}{x-y}$, &c.

$$\text{Here } a=x^2-y^2, \text{ and } r=\frac{x+y}{x^2-y^2}=\frac{1}{x-y},$$
$$\therefore\quad u=(x^2-y^2)\cdot\frac{1}{(x-y)^{n-1}}=\frac{(x+y)(x-y)}{(x-y)^{n-1}}=\frac{x+y}{(x-y)^{n-2}};$$
$$s=a\left(\frac{r^n-1}{r-1}\right)=(x^2-y^2)\left\{\frac{\frac{1}{(x-u\,^{n}}-1}{\frac{1}{(x-y)}-1}\right\}=\frac{x+y}{(x-y)^{n-2}}\cdot\left\{\frac{1-(x-y)^n}{1-(x-y)}\right\},$$

or $\frac{x+y}{(x-y)^{n-2}}\cdot\left\{\frac{(x-y)^n-1}{(x-y)-1}\right\}$.

Exercise LXI.

Find the last term and the sum of the terms:

1. $1+2+4+$ &c. to 7 terms. 2. $4+12+36+$ &c. to 10 terms.
3. $5+20+80+$ &c. to 9 terms. 4. $9+15\frac{3}{4}+28\frac{1}{4}+$ &c. to 7 terms.
5. $6\frac{1}{4}+9\frac{3}{8}+15\frac{5}{16}+$ &c. to 8 terms. 6. $6+4\frac{1}{2}+3\frac{3}{8}+$ &c. to 6 terms.

Find the sum of the series:

7. $a=8$, $r=\frac{1}{2}$, to 15 terms.
8. $a=\frac{5}{6}$, $r=\frac{2}{3}$, to 11 terms.
9. $a=3$, $r=\frac{7}{8}$, to 25 terms.
10. $a=3\frac{1}{2}$, $r=\frac{3}{8}$, to 8 terms.
11. $a=7\frac{1}{2}$, $r=\frac{27}{20}$, to 31 terms.
12. $a=5560$, $r=\frac{9}{11}$, to 40 terms.

When r is a proper fraction, the formula (2) may be conveniently written

$$s=\frac{a(1-r^n)}{1-r}=\frac{a}{1-r}-\frac{ar^n}{1-r}.$$

Now, if the ratio r is less than 1, r^n will decrease as n increases; and if we suppose n to increase *indefinitely*, r^n will become *indefinitely* small, constantly approximating, but never actually becoming 0. Hence we see that if we have a geometrical series whose ratio is less than unity, there is a quantity $\frac{a}{1-r}$ to which the sum of the series constantly approaches as n increases; and therefore we say that the sum of such a series continued *ad infinitum* is $\frac{a}{1-r}$. This expression is termed the *limit* of the sum of the series.

Ex. 1. Find the limit of the sum of the series $1+\frac{1}{4}+\frac{1}{16}$, &c.

$$a=1,\ r=\frac{1}{4},\ \therefore\ \Sigma=\frac{1}{1-\frac{1}{4}}=\frac{1}{\frac{3}{4}}=1\tfrac{1}{3}.$$

Ex. 2. Find the limit of the sum of the series $11+2\frac{4}{9}+\frac{44}{81}$, &c.

$$\text{Here } a=11, \text{ and } r=\frac{2}{9},\ \therefore\ \Sigma=\frac{11}{1-\frac{2}{9}}=\frac{\frac{11}{1}}{\frac{7}{9}}=\frac{99}{7}=14\tfrac{1}{7}.$$

A familiar example of a geometrical series continued *ad infinitum* occurs in the case of a recurring decimal.

Ex. 3. The recurring decimal $\cdot 222\ldots=\frac{2}{10}+\frac{2}{10^2}+\frac{2}{10^3}\ldots$, and the sum of the series is $\frac{\frac{2}{10}}{1-\frac{1}{10}}=\frac{2}{9}$; that is, the more terms we take of this series, the more nearly will their sum be equal to $\frac{2}{9}$.

Exercise LXII.

Sum up the following series *ad infinitum:*

1. $1-\frac{1}{2}+\frac{1}{4}-$ &c.
2. $8+4+2+$ &c.
3. $\frac{2}{3}-\frac{2}{9}+\frac{2}{27}-$ &c.
4. $1+\frac{1}{2}+\frac{1}{4}+$ &c.
5. $40+17\frac{1}{7}+7\frac{17}{49}+$ &c.
6. $9+6+4+$ &c.
7. $3-\frac{2}{5}+\frac{4}{75}-$ &c.
8. $2-1\frac{1}{2}+1\frac{1}{8}-$ &c.
9. $\frac{1}{3}-\frac{1}{3.2}+\frac{1}{3.2^2}-$ &c.
10. $-\frac{2}{3}+\frac{1}{3}-\frac{1}{6}+$ &c.
11. $\frac{3}{10}+\frac{3}{10^2}+\frac{3}{10^3}+$ &c.
12. $\frac{2}{10}-\frac{2}{10^2}+\frac{2}{10^3}-$ &c.
13. $\cdot 323232\ldots\ldots$
14. $\cdot 232323\ldots\ldots$

3. Given the first term, the last term, and the sum of a geometrical series, to find the ratio and the number of terms.

In the fundamental equations $u=ar^{n-1}$, $s=\frac{ru-a}{r-1}$, a, u, s are given to find r and n. From the second we obtain $r=\frac{s-a}{s-u}$. And to find n we obtain from both equations $r^{n-1}=\frac{u}{a}$, $r=\frac{s-a}{s-u}$. But n being in the form of an index, logarithms must be employed to determine its value. We shall have

$$\log. r=\frac{\log. u-\log. a}{n-1},$$

$$\log. r=\log.(s-a)-\log.(s-u),$$

$\therefore\ \frac{\log. u-\log. a}{n-1}=\log.(s-a)-\log.(s-u)$, an equation in which n is the only unknown quantity. The equation gives $n=1+\frac{\log. u-\log. a}{\log.(s-a)-\log.(s-u)}$.

4. Given the number of terms, the last term, and the sum of a geometrical progression, to find the first term and the ratio.

In equations $u=ar^{n-1}$, $s=\frac{ru-a}{r-1}$, n, u, s are known, r and a unknown. To find r determine a from both equations.

$$a=\frac{u}{r^{n-1}},\quad a=ru-(r-1)s.$$

Equating these, $\frac{u}{r^{n-1}}=ru-(r-1)s,$

$$u=r^nu-r^{n-1}(r-1)s,$$

$$(s-u)r^n-sr^{n-1}+u=0,$$

$$r^n-\frac{s}{s-u}r^{n-1}+\frac{u}{s-u}=0.$$

The value of r will then be determined by an equation of the n^{th} degree.

To find a value r in both the foregoing equations.

$$r=\sqrt[n-1]{\frac{u}{a}},\quad r=\frac{s-a}{s-u},$$

$$r^{n-1}=\frac{u}{a},\quad r^{n-1}=\left(\frac{s-a}{s-u}\right)^{n-1},$$

$$\frac{u}{a}=\left(\frac{s-a}{s-u}\right)^{n-1},$$

$$\therefore\ a(s-a)^{n-1}-u(s-u)^{n-1}=0,$$

in which r again may be found by an equation of the n^{th} degree.

The following table shows the solution of every case to be met with in geometrical progression.

Given.	Required.	
$a,\ r,\ n,$	u	$u=ar^{n-1}.$
$a,\ r,\ s,$		$u=\frac{a+(r-1)s}{r}.$
$a,\ n,\ s,$		$u(s-u)^{n-1}-a(s-a)^{n-1}=0.$
$r,\ n,\ s,$		$u=\frac{(r-1)sr^{n-1}}{r^n-1}.$

Given.	Required.	
a, r, n,	s	$s=\dfrac{a(r^n-1)}{r-1}$.
a, r, u,		$s=\dfrac{ru-a}{r-1}$.
a, n, u,		$s=\dfrac{\sqrt[n-1]{u^n}-\sqrt[n-1]{a^n}}{\sqrt[n-1]{u}-\sqrt[n-1]{a}}$.
r, n, u,		$s=\dfrac{u(r^n-1)}{(r-1)r^{n-1}}$.
r, n, u,	a	$a=\dfrac{u}{r^{n-1}}$.
r, n, s,		$a=\dfrac{(r-1)s}{r^n-1}$.
r, u, s,		$a=ru-(r-1)s$.
n, u, s,		$a(s-a)^{n-1}-u(s-u)^{n-1}=0$.
a, n, u,	r	$r=\sqrt[n-1]{\left(\dfrac{u}{a}\right)}$.
a, n, s,		$r^n-\dfrac{s}{a}r+\dfrac{s-a}{a}=0$.
a, u, s,		$r=\dfrac{s-a}{s-u}$.
n, u, s,		$r^n-\dfrac{s}{s-u}r^{n-1}+\dfrac{u}{s-u}=0$.
a, r, u	n	$n=\dfrac{\log. u-\log. a}{\log. a}+1$.
a, r, s,		$n=\dfrac{\log. \{a+(r-1)s\}-\log. a}{\log. r}$.
a, u, s,		$n=\dfrac{\log. u-\log. a}{\log. (s-a)-\log. (s-u)}+1$.
r, u, s,		$n=\dfrac{\log. u-\log. \{ru-(r-1)s\}}{\log. r}$.

Exercise LXIII.

1. How many terms of the series $1+2+4+$ &c. must be taken to make 8191?

2. The sum of the first and second of four numbers in geometrical progression is 15, and the sum of the third and fourth 240. What are the numbers?

3. In a geometrical progression of four terms the sum of the terms is a, and the sum of their squares b. What is the progression?

4. The fifth term of a geometrical series is 8 times the second, and the sum of the first and third 15. What is the series?

5. If a be the first term, and b the sum of 3 terms of a geometrical progression, what is the common ratio?

6. Insert four geometrical means between 2 and 64, and four between $\frac{1}{9}$ and 27.

7. Place 11 terms in geometrical progression between 1 and $\frac{1}{2}$.

8. What ratio has a geometrical progression of 32 terms, whose first term is 5, and last term 80? What is the 20th term of the progression?

9. The sum of an infinite series is $\frac{2}{3}$, and the sum of the first two terms is $\frac{14}{25}$. What is the series?

10. If four quantities are in geometrical progression, show that the sum of the extremes is greater than the sum of the means.

HARMONICAL PROGRESSION.

Three quantities are said to be in harmonical progression when the first term is to the third as the difference between the first and second is to the difference between the second and third.

a, b, c are in Har. Prog. when $a : c = a-b : b-c$. Four quantities are said to be in Har. Prog. when the first term is to the fourth term as the difference between the first and second is to the difference between the third and fourth.

a, b, c, d are in Har. Prog. when $a : d = (b-a) : (d-c)$. Any series of quantities is in Har. Prog. when any consecutive three are in Har. Prog.

Hence it follows that in three harmonic progressionals a, b, c, any two of them being given, the third may be found. a, c being given, to find b.

$$a : c = a-b : b-c.$$
$$ab-ac = ac-bc,$$
$$\therefore\ b(a+c) = 2ac,$$
$$\therefore\ b = \frac{2ac}{a+c};$$

i. e. the harmonical mean between two quantities is equal to twice their product divided by their sum. And from the same equation we obtain (a and b being given) c the third harmonic proportional, $c = \frac{ab}{2a-b}$.

Let a, b, c be in Har. Prog.; then, by definition, $a : c = a-b : b-c$, and $ab-ac = ac-bc$. Dividing this by abc, we get $\frac{1}{c} - \frac{1}{b} = \frac{1}{b} - \frac{1}{a}$; *i.e.* the difference between $\frac{1}{a}$ and $\frac{1}{b}$ is equal to the difference between $\frac{1}{b}$ and $\frac{1}{c}$; $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are therefore in arithmetical progression.

Hence we may take, as an adequate definition of harmonics:—quantities are in harmonical progression whose reciprocals are in arithmetical progression.

This property enables us to perform many problems with harmonical series. Thus, to find harmonical means, to continue a series, &c. we take the reciprocals of the given quantities, and deal with them as with quantities in arithmetical progression.

Ex. 1. Insert two harmonical means between 3 and 12.

Inverting, $\frac{1}{3}$ is first term and $\frac{1}{12}$ last term of an arith. series of 4 terms. The common difference is $-\frac{1}{12}$. The arith. means will be $\frac{1}{4}$ and $\frac{1}{6}$. Hence the har. means are 4 and 6.

Ex. 2. Continue $\frac{1}{4}$, $\frac{1}{6}$, $\frac{1}{8}$ to 3 terms each way.

The reciprocals are 4, 6, 8, an Arith. Prog. with common difference 2. The series will be $-2, 0, 2, 4, 6, 8, 10, 12, 14$. Hence the harmonics will be $-\frac{1}{2}$, ∞, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{6}$, $\frac{1}{8}$, $\frac{1}{10}$, $\frac{1}{12}$, $\frac{1}{14}$.

EXERCISE LXIV.

1. Continue to 3 terms each way the series 2, $2\frac{2}{3}$, 4; $1\frac{1}{2}$, 2, 3; $\frac{3}{23}$, $\frac{3}{20}$, $\frac{3}{17}$.

2. Insert three harmonical means between 1 and $\frac{1}{9}$, and three between $1\frac{3}{5}$ and 8.

3. Find a Har. third to 2 and 3, and a Har. fourth to 6, 4, 3.

The arithmetical mean A between a and c is $\frac{a+c}{2}$; for, by definition, $A-a=c-A$, $\therefore 2A=a+c$, $\therefore A=\frac{a+c}{2}$.

The geometrical mean $G=\sqrt{ac}$, for $\frac{G}{a}=\frac{c}{G}$, $\therefore G^2=ac$, $\therefore G=\sqrt{ac}$.

The harmonical mean H is $\frac{2ac}{a+c}$, for $a:c=a-H:H-c$, $\therefore H=\frac{2ac}{a+c}$.

A, G, H are in geometrical progression, for $\frac{a+c}{2}:\sqrt{ac}=\sqrt{ac}:\frac{2ac}{a+c}$.

If A, G, H are the Arith., Geom., and Har. means between the same two quantities a and c, then A is $> G$, and $G > H$.

$A=\frac{a+c}{2}$, $G=\sqrt{ac}$; $\therefore$ if A is $> G$, then $a+c > 2\sqrt{ac}$, and $a^2+2ac+c^2 > 4ac$, $\therefore a^2-2ac+c^2=(a-c)^2 > 0$, $\therefore A$ is $> G$; and in the same manner it may be shown that $G > H$.

The three kinds of progression which have been examined may here be exhibited under one view.

If a, b, c are in arithmetical progression, $\frac{a-b}{b-c}=\frac{a}{a}$.

If a, b, c are in geometrical progression, $\frac{a-b}{b-c}=\frac{a}{b}$.

If a, b, c are in harmonical progression, $\frac{a-b}{b-c}=\frac{a}{c}$.

EXERCISE LXV.

1. Find Arith., Geom., and Har. means between 8 and $\frac{1}{2}$.
2. Find Arith., Geom., and Har. means between 18 and 2.
3. What are the Arith., Geom., and Har. means between $2\frac{1}{4}$ and 4?
4. Find a harmonical third to 5 and 8.
5. Find a harmonical mean between -4 and 18.
6. Find a harmonical first to 12 and 7.
7. The sum of three terms of a harmonical series is $1\frac{1}{12}$, and the first term is $\frac{1}{2}$. What is the series?
8. The arithmetical mean between two numbers exceeds the geometrical by 13, and the geometrical exceeds the harmonical by 12. Determine the numbers.

CHAPTER XV.

RATIOS.

RATIO is the relation which quantities of the same kind bear to one another in respect of magnitude.

Thus, if one magnitude is $\frac{3}{4}$ of another magnitude, they are said to be in the ratio of 3 to 4. For if both the magnitudes be divided into equal parts, the former will contain three and the latter four of these parts. The expressions 3 : 4 and $\frac{3}{4}$ both convey the same idea. $\frac{3}{4}$ denotes that unity has been divided into four parts, and three of those parts taken. Therefore $\frac{3}{4} : 1$ or $\frac{3}{4}$ as $3 : 4$; $a : b$ and $\frac{a}{b}$ equally express a ratio, the fraction $\frac{a}{b}$ denoting the multiple or part that a is of b.

The former term of a ratio is called the *antecedent*, and the latter the *consequent*. When a ratio is represented fractionally, the antecedent is the numerator, and the consequent the denominator.

The quantity by which the antecedent must be multiplied in order to produce the consequent is called the *exponent* of the ratio.

If a is $>$ b in the ratio $\frac{a}{b}$, $\frac{a}{b}$ is called a ratio of *greater inequality*.

If $a=b$, $\frac{a}{b}$ is termed a ratio of *equality*.

When $a < b$, $\frac{a}{b}$ is a ratio of *less inequality*.

The terms of a ratio may be multiplied or divided by the same number without affecting the value of the ratio.

$$\text{Thus } \frac{a}{b}=\frac{3a}{3b},\ \frac{am}{bm}=\frac{a}{b}.$$

If to both terms of a ratio $a : b$ the same quantity be added, the ratio will be diminished or increased according as $\frac{a}{b}$ is a ratio of less or greater inequality.

Let $\frac{a}{b}$ be a ratio, and let the quantity x be added to both terms, then $\frac{a+x}{b+x} >$ or $< \frac{a}{b}$, according as

$$\begin{aligned} & \frac{b(a+x)}{b(b+x)} > \text{ or } < \frac{a(b+x)}{b(b+x)}, \\ \text{or as } \quad & b(a+x) > \text{ or } < a(b+x), \\ \text{or as } \quad & bx > \text{ or } < ax, \\ \text{or as } \quad & b > \text{ or } < a; \end{aligned}$$

that is, according as the ratio is one of less or greater inequality.

Again, suppose to both terms of the ratio 4 : 9, 2 be added, the ratio will then be as 6 : 11, and $\frac{6}{11}$ or $\frac{54}{99} > \frac{4}{9}$ or $\frac{44}{99}$. But if 2 be added to both terms of 9 : 4, then $\frac{11}{6}$ or $\frac{66}{36} < \frac{9}{4}$ or $\frac{81}{36}$.

Problems on ratios are solved by expressing the ratios in fractional form, and treating them by the ordinary rules of fractions. Thus, ratios are compared by reducing them to a common denominator, and comparing their numerator. They are reduced to their *lowest terms* by dividing both the antecedent and the consequent by their greatest common measure, and they are *compounded* by multiplying them together.

The *duplicate* ratio of two quantities is the ratio of their squares, as $a^2 : b^2$.

The *triplicate* ratio is the ratio of their cubes, as $a^3 : b^3$.

The *quadruplicate* ratio is the ratio of their fourth powers, as $a^4 : b^4$.

The *subduplicate* ratio is the ratio of their square roots, as $a^{\frac{1}{2}} : b^{\frac{1}{2}}$.

EXERCISE LXVI.

1. Compare the ratios $5 : 6$ and $4 : 7$, $11 : 27$ and $17 : 24$, $7 : 16$ and $11 : 18$.

2. Which is greater of the ratios $a+b : b$ and $3a : ab$, $2a+1 : 3a+2$ and $4a+5 : 5a+4$, $3x+2 : x-1$ and $5x-3 : x+1$, $1 : x+1$ and $x+1 : x^2-x+1$.

3. Reduce to their lowest terms the ratios $1296 : 1728$, $990 : 1935$, $324 : 396$.

4. Compound the ratios $12 : 13$, $8 : 3$, $26 : 25$, $75 : 64$; $32 : 9$, $7 : 4$, $3 : 8$, and $7 : 15$; $7 : 11$, $5 : 2$, $3 : 7$, and $21 : 2$.

5. Compound the ratios $a^2+x^2 : ax$, $b^2+y^2 : by$, $x^2-a^2 : ax$, and $y^2-b^2 : by$.

6. Compound the ratios $a^2-3ax+x^2 : a^2-x^2$ and $(a-x)(a+3x) : a-3x$.

7. Compound the duplicate ratios of $3 : 4$, the triplicate of $2 : 3$, the subduplicate of $16 : 81$, and the subtriplicate of $64 : 343$.

8. Show that the ratio $a+b : a-b$ is increased by subtracting x from both terms.

9. Compare the areas of two similar triangles whose bases are 77 and 91 feet respectively. (The areas of similar superficies are (Euc. vi. 19, 20) in the duplicate ratio of their homologous sides.)

10. Compare the areas of two spheres whose radii are $8\frac{1}{2}$ and 8 inches respectively. (Similar solids are to each other as the cubes of their like dimensions.)

11. Jupiter is about 11 years in performing his revolution round the Sun: find his mean distance. (The time of a planet's revolution is in the sesquiplicate ratio $[a^{\frac{3}{2}} : b^{\frac{3}{2}}]$ of its mean distance.)

12. The mean distance of Saturn from the Sun is 900 millions of miles. In what time does he complete his orbit?

PROPORTION.

Proportion may be defined as the *equality of ratios*. When two ratios, as $a:b$ and $c:d$, are equal, that is, $a:b=c:d$, or $\frac{a}{b}=\frac{c}{d}$, a is said to have to b the same ratio that c has to d; or a is the same multiple or part of b that c is of d. The proportion $a:b=c:d$ is read a *is to* b *as* c *is to* d. a and d are called the *extremes*, b and c the *means*.

In any proportion the product of the extremes is equal to the product of the means.

For if $a:b=c:d$, then $\frac{b}{a}=\frac{d}{c}$, and multiplying both sides by ac, $\frac{bac}{a}=\frac{adc}{c}$, $\therefore bc=ad$.

And conversely, if the product of two quantities is equal to the product of two other quantities, the four are proportional.

For (by hyp.) $ad=bc$, and dividing both sides by bd, we obtain $\frac{ad}{bd}=\frac{bc}{bd}$, $\therefore \frac{a}{b}=\frac{c}{d}$, $\therefore a:b=c:d$.

To find a fourth proportional to three given quantities a, b, c.

Let the fourth proportional be x, then

$$a:b=c:x,$$

$$\text{and}\quad ax=bc,\ \therefore x=\frac{bc}{a},$$

and this is the algebraical expression for the Rule of Three in arithmetic, which is: Multiply the second and third terms together, and divide by the first.

To find a third proportional to two given quantities a, b.

As before, let $x=$ the third proportional, then

$$a:b=b:x,$$

$$ax=b^2,\ \therefore x=\frac{b^2}{a},$$

i.e. the third proportional is equal to the square of the second term divided by the first.

To find a mean proportional between two given quantities a, b.

Let $x=$ mean proportional,

then $a:x=x:b$, $\therefore ab=x^2$, $\therefore x=\sqrt{ab}$,

i.e. the square root of the product of two quantities is the mean proportional between them.

$a:x=x:b$ is called a continued proportion.

From $a:b=c:d$ there may be formed eight different proportions, by transposing the terms, thus:

$a:b=c:d$, (1)

by alternation of (1) $a:c=b:d$, (2)

by inversion of (1) $b:a=d:c$, (3)

by alternation of (3) $b:d=a:c$, (4)

by inversion of (2) $c:a=d:b$, (5)
by equal ratios in (1) $c:d=a:b$, (6)
by inversion of (4) $d:b=c:a$, (7)
by alternation of (7) $d:c=b:a$. (8)

If $a:b=c:d$, then

1. $ma:mb=nc:nd$.
2. $ma:nb=mc:nd$.
3. $\frac{a}{m}:\frac{b}{m}=\frac{c}{n}:\frac{d}{n}$.
4. $\frac{a}{m}:\frac{b}{n}=\frac{c}{m}:\frac{d}{n}$.

1. Because $a:b=c:d$, $\frac{b}{a}=\frac{d}{c}$, and $\frac{mb}{ma}=\frac{nd}{nc}$,

$\therefore\ ma:mb=nc:nd$.

2. Because $\frac{b}{a}=\frac{d}{c}$, and $\frac{n}{m}=\frac{n}{m}$, $\therefore\ \frac{b}{a}\cdot\frac{n}{m}=\frac{d}{c}\cdot\frac{n}{m}$, that is, $\frac{nb}{ma}=\frac{nd}{mc}$,

$\therefore\ ma:nb=mc:nd$.

The third and fourth of the above propositions follow from the first and second respectively, by substituting in (1) $\frac{1}{m}$ for m, and in (2) $\frac{1}{n}$ for n.

If $a:b=c:d$, and $a \gtrless b$, then $c \gtrless d$.

Because $a:b=c:d$, $\therefore\ \frac{b}{a}=\frac{d}{c}$. Now, according as $a \gtrless b$, $\frac{b}{a}$ and $\therefore\ \frac{d}{c}$ are proper or improper fractions, *i.e.* $c \gtrless d$. In like manner is it seen that if $a:b=c:d$, and $a \gtrless c$, then also $b \gtrless d$.

Euclid's definition of proportion in his fifth book is: Four magnitudes are said to be proportional when any equimultiples whatever of the first and third, and any whatever of the second and fourth, being taken, if the multiple of the first is $> = <$ than that of the second, the multiple of the third is $> = <$ than that of the fourth. That the algebraical representation of proportion is in accordance with this definition is very evident.

For if $a:b=c:d$, then $\frac{a}{b}=\frac{c}{d}$, and $\frac{ma}{nb}=\frac{mc}{nd}$.

Now, as $ma > = < nb$, so $mc > = < nd$.

And conversely, from the same definition may be deduced the algebraical rule of proportion $\frac{a}{b}=\frac{c}{d}$.

For let a, b, c, d be four quantities such that any equimultiples ma, mc of the first and third being taken, and any equimultiples nb, nd of the second and fourth, if $ma > nb$, then $mc > nd$, if equal, equal, and if less, less.

Now since we may make m, n of any value we please, we can take them so that $ma=nb$, and consequently $mc=nd$; $\therefore\ \frac{ma}{nb}=\frac{mc}{nd}$, $\therefore\ \frac{a}{b}=\frac{c}{d}$.

If $a : b = c : d$, then (1) $a \pm b : c \pm d = a : c = b : d$.
(2) $a \pm c : b \pm d = a : b = c : d$.

Because (1) $a : b = c : d$, and $ad = bc$,

$$\therefore \quad ad \pm bd = bc \pm bd,$$
$$\text{or} \quad (a \pm b)d = (c \pm d)b.$$

Also because $a \pm b : c \pm d = b : d$,

and $a : c = b : d$,

$$\therefore \quad a \pm b : c \pm d = a : c = b : d.$$

Because (2) $a : b = c : d$, and $a : c = b : d$, $\therefore$ it follows by a process similar to that in (1) that $a \pm c : b \pm d = a : b = c : d$.

If $a : b = a' : b' = a'' : b'' = \dots\dots a^{(n)} : b^{(n)}$,

then $a + a' + a'' + \dots\dots + a^{(n)} : b + b' + b'' + \dots\dots + b^{(n)} = a : b$.

That is, if any quantities be in continued proportion, the sum of all the antecedents is to the sum of all the consequents as one of the antecedents is to its consequent.

By the last proposition,

$$a + a' : b + b' = a' : b' = a'' : b'',$$
$$a + a' + a'' : b + b' + b'' = a'' : b'' = a''' : b''',$$
$$a + a' + a'' + a''' : b + b' + b'' + b''' = a''' : b''' = a'''' : b'''',$$
$$\text{and} \therefore a + a' + a'' + \dots + a^{(n)} : b + b' + b'' + \dots + b^{(n)} = a^{(n)} : b^{(n)}.$$

But $a : b = a^{(n)} : b^{(n)}$,

$$\therefore \quad a + a' + a'' + \dots a^{(n)} : b + b' + b'' + \dots b^{(n)} = a : b.$$

If $a : b = \alpha : \beta$, and $a' : b' = \alpha' : \beta'$, and $a'' : b'' = \alpha'' : \beta''$,

then $aa'a'' : bb'b'' = \alpha\alpha'\alpha'' : \beta\beta'\beta''$.

$$\frac{a}{b} = \frac{\alpha}{\beta}, \text{ and } \frac{a'}{b'} = \frac{\alpha'}{\beta'}, \text{ and } \frac{a''}{b''} = \frac{\alpha''}{\beta''}.$$

By multiplication, $\frac{aa'a''}{bb'b''} = \frac{\alpha\alpha'\alpha''}{\beta\beta'\beta''}$, $\therefore aa'a'' \dots : bb'b'' \dots = \alpha\alpha'\alpha'' \dots : \beta\beta'\beta'' \dots$

This is called *compounding* the proportions.

EXERCISE LXVII.

1. Find a third proportional to 3, 6 ; 2, 7 ; 4, 9 ; $\frac{2}{3}$, $\frac{3}{4}$.
2. Find a fourth proportional to 2, 3, 4 ; 7, 6, 12 ; $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{4}$.
3. Find a mean proportional between 8, $12\frac{1}{2}$; 3, 27 ; $4\frac{1}{2}$, $\frac{1}{2}$.
4. Prove the truth or otherwise of the following proportions :

   (i.) $3a + 4b : 9a + 8b = a - 2b : 3a - 4b$.

   (ii.) $9a^2 - 4b^2 : 15a^2 - 31ab + 14b^2 = 15a^2 + 31ab + 14b^2 : 25a^2 - 49b^2$.

   (iii.) $a^3 + b^3 : a + b = a^5 - a^4b + a^3b^2 - a^2b^3 + ab^4 - b^5 : a^3 - b^3$.

   (iv.) $a^3 + b^3 : a^2 + b^2 = a^2 - b^2 : a - b$.

5. Why is a^2-b^2 the mean proportional between $a^2+2ab+b^2$ and $a^2-2ab+b^2$?

6. If $a:b=c:d$, why is $ma\pm nb:pa\pm qb=mc\pm nd$?

Solve the equations:

7. $(x+2):(20-x)=(x+20):(46-x)$.

8. $\left.\begin{array}{l}(5x+7y):(9x+33)=31:21,\\(11x+27):(7x+5y)=19:11.\end{array}\right\}$

9. $\left.\begin{array}{l}(ax+by):(cx+d)=m:n,\\(e+fy):(gx+hy)=p:q.\end{array}\right\}$

10. $x:y=a+b-\dfrac{ab}{a+b}:a-b+\dfrac{ab}{a-b}$, when $x+y=2a^3$.

11. $x:y:z:u=a^3+a^2+a+1:a^2+a+1:a+1:1$. What are the values of x, y, z, u?

12. Divide 266 in the proportion of 1 : 2 : 4.

13. A boat which had started from a certain place 10 days, is pursued by another from the same place. The rate of sailing of the former is to that of the latter as 4 : 9. When will the latter overtake the former?

14. Two bodies start from the same point, the second n seconds later than the first, and its speed is to that of the first as $q:p$. In what time will they be together?

15. My age is now 20, my brother's 30. In how many years will our ages be as 4 : 5?

16. A hare is 50 leaps before a greyhound, and takes 4 leaps to the greyhound's 3; but 2 of the greyhound's leaps are as much as 3 of the hare's. How many leaps must the greyhound take to catch the hare?

VARIATION.

When one quantity y depends upon another x, so that if x is changed in value there is a proportionate change in the value of y, then y is said *to vary as* x.

The symbol $\propto$ is employed to denote variation. We have an illustration of variation in Euc. vi. 1. *Triangles and parallelograms of the same altitude are to one another as their bases.* If we double the base of a triangle or parallelogram, their altitude remaining the same, we double their area; and in proportion as we alter the base, in the same proportion do we alter the area. Given the altitude, the area therefore is said to vary as the base.

If $Y\propto X$, then the ratio $Y:X$ will remain unaltered whatever may be the values of these quantities; Y will be to X in a constant ratio, and if we denote this constant value by μ, the expression $Y\propto X$ will be equivalent to $Y=\mu X$, for $\dfrac{Y}{X}=\mu$.

If $Y=\mu X$, Y is said to vary *directly* as X, Y increasing or decreasing as X increases or decreases. And if y and x be a pair of values of Y and X, then since $Y:X=\mu$, and $y:x=\mu$, $\therefore$ $Y:X=y:x$, or $Y:y=X:x$; that is, if any pair of values of Y and X be taken, the four will be proportional.

If $y=\frac{\mu}{x}$, y is said to vary *inversely* as x, y decreasing or increasing as x increases or decreases.

If $y=\mu xz$, y is said to vary *jointly* as xz.

If $y=\frac{\mu x}{z}$, y is said to vary *directly* as x, and *inversely* as z.

If $y \propto x$, and $x \propto z$, then $y \propto z$.

For let $y=\mu x$, and $x=\mu' z$,

then $y=\mu\mu' z$. But $\mu\mu'$ is constant, and therefore $y \propto z$.

If $y \propto x$, and $x \propto z$, then $x \propto \sqrt{yz}$.

$$\text{Let } y=\mu x, \; z=\mu' x,$$

then $yz=\mu\mu' x^2$, $\therefore \sqrt{yz}=x\sqrt{\mu\mu'}$, and $\sqrt{\mu\mu'}$ being constant, $\therefore x \propto \sqrt{yz}$.

If $y \propto x$, and 3 and 5 are contemporaneous values of y and x, we may determine the value of μ, and consequently express y in terms of x.

$$y=\mu x,$$
$$\therefore \; 3=\mu 5,$$
$$\therefore \; \tfrac{3}{5}=\mu, \; \therefore \; y=\tfrac{3}{5}x.$$

If x, y, z are quantities depending upon one another in such a manner that $y \propto x$ when z is constant, and $y \propto z$ when x is constant, then $y \propto xz$ when x and z both vary.

Let $y=\mu xz$, in which μ is a constant quantity which may involve either x or z. When x is constant, $y \propto z$; but $y=\mu xz$, $\therefore \mu x$ does not involve z, nor does μ involve z. Again, when z is constant, $y \propto x$; but $y=\mu zx$, $\therefore \mu z$ does not involve x, nor does μ involve x, $\therefore \mu$ is constant, and $y \propto xz$.

As an illustration of this proposition we may instance the case of a triangle. Let $x=$ the altitude, z the base, and y the area of a triangle; then when x is given $y \propto z$, and when z is given $y \propto x$, but when neither of these is given $y \propto xz$.

That this proposition is the foundation of the Rule of Three in arithmetic the student will readily perceive. For suppose that a certain capital A placed out at interest a certain time T produces a certain interest I, it is clear that the interest will vary according as the capital or time varies,—that I in fact will $\propto AT$; and if i be the amount of interest arising from capital a during time t, then

$$I : i = AT : at.$$

Ex. If 240*l.* in 16 months gain 64*l.*, how much will 60*l.* gain in 6 months?

Here $A=240l.$, $a=60l.$, $T=16$ months, $t=6$ months, $I=64l.$, $i=x$; and $AT : at :: I : x$.

$$\text{Therefore } \; 240\times 16 : 60\times 6 :: 64 : x,$$
$$\therefore \; x\times 240\times 16=60\times 6\times 64;$$
$$\therefore \; x=6l.$$

EXERCISE LXVIII.

1. If $y \propto x$, and when $x=2$, $y=3$, express y in terms of x.

2. If $y \propto \frac{1}{x}$, and when $x=\frac{1}{4}$, $y=16$, express y in terms of x.

3. If $x \propto yz^2$, and 2, 3, 4 be the values of x, y, z, express x in terms of y and z.

4. If $2y+3z \propto 4y+5z$, show that $y \propto z$.

5. If $x=$ the sum of two quantities, which vary as y, y^2, and when $y=2$, $x=8$, when $y=3$, $x=15$: express x in terms of y.

6. If $x=$ the sum of three quantities, the first being constant, the second varying as y, and the third as y^2, and when y has values 3, 4, 5, $x=8$, 12, 20 : express x in terms of y.

CHAPTER XVI.

PERMUTATIONS, COMBINATIONS, AND VARIATIONS.

PERMUTATION means the different ways in which the order or relative position of any given number of things may be changed. The only object to be regarded in permutation is the *order* in which the things are placed, for no two arrangements are to have all the quantities in the same relative position.

For example, two things, *a* and *b*, are capable of only two changes in their relative position, *ab*, *ba*, and this number is expressed by 1 × 2 ; but three things, *a*, *b*, *c*, are capable of six variations, viz. *abc*, *acb*, *bac*, *bca*, *cab*, *cba*, and this number of permutations is expressed by 1 × 2 × 3.

Four things are capable of twenty-four variations, viz.

abcd
abdc
acbd
acdb
adbc
adcb
bacd
badc
bcad
bcda
bdac
bdca
cabd
cadb
cbad
cbda
cdab
cdba
dabc
dacb
dbac
dbca
dcab
dcba

and this number of permutations is expressed by 1 × 2 × 3 × 4. In like manner, when there are five things, four of them (leaving out the fifth) will have 24 variations, consequently, by taking in the fifth, there will be 5 times 24 variations.

Permutation and variation are terms used synonymously, though properly speaking there is a distinction between them. Permutation signifies the different arrangements of the terms taken *all* together, Variation the different arrangements that can be made of them taking a *certain number* together.

To find the number of variations of n things a, b, c, d, &c. taken 2 together.

If nVr denote the number of variations of n things taken r together, it is manifest that the number of variations of n things taken 1 together is n, that is, $nV1=n$. But let us remove a, and there will remain $n-1$ things, b, c, d, e, &c.; and if we place a before each of these, it is obvious that we shall have $(n-1)$ variations in which a stands first. Again, if we remove b, there will remain $(n-1)$ things, and consequently $(n-1)$ variations in which b stands first. In like manner of the rest. And as there are n things, each of which may stand first, so there will be $n(n-1)$ variations.

$$nV2=n(n-1).$$

The number of variations of n things a, b, c, d, e, &c. taken *three* together, is found in a similar manner. First, leave out a as before, then the number of variations of the remaining $(n-1)$ things *taken two together* will be $(n-1)(n-2)$; and now, if a be placed before each of these, there will be $(n-1)(n-2)$ variations of n things taken three together, a standing first. So likewise there will be $(n-1)(n-2)$ variations in which b stands first, a like number in which c stands first, and so on for each of the n things; and therefore the whole number of variations of n things taken three together, or $nV3=n(n-1)(n-2)$.

And we infer that $nV4=n(n-1)(n-2)(n-3)$,

$$nV5=n(n-1)(n-2)(n-3)(n-4),$$

and generally that the number of variations of n things taken r together,

$$\text{or } nVr=n(n-1)(n-2)(n-3)(n-4)\ldots\ldots\{n-(r-1)\}.$$

It is observed that the negative number of the last factor is always unity less than the number of things taken together.

Hence we may find the number of permutations of n things. For r in such case will equal n, and the last factor in the formula above will become $n-(n-1)=1$,

$$\text{and } nVn=n(n-1)(n-2)(n-3)\ldots\ldots 2.1.$$

Ex. How many permutations may be made of the letters of the word *number?*

$$6.\ 5.\ 4.\ 3.\ 2.\ 1=720.$$

In the preceding remarks we have supposed that the things were *different*; but when several of them are of one kind, and several of another, it is obvious that the permutations that can be made of them will not be so great as when all the things are different. For instance, we have seen that the letters a, b, c admit of six variations; but if two of the quantities be alike, as *aab*, the six variations will be reduced to three, *aab*, *baa*, *aba*. We have seen also that the letters a, b, c, d admit of 24 variations; but if we have *aabb*, the 24 variations are reduced to 6, viz. *aabb*, *abba*, *abab*, *bbaa*, *baab*, *baba*.

Therefore to find the number of permutations of n things taken all together, whereof α things are of one kind, β of another, and γ of another:

Let x be the number of permutations.

Now, since all the α like things enter into each permutation, if we suppose them different they would admit of $\alpha(\alpha-1)\ldots 2.1$ permutations, and consequently the whole number of permutations of n things would then be $\alpha \ldots . 2.1 \times x$.

So also, since β like things enter into each permutation, if we suppose them to be *different* we shall obtain in like manner $\beta \ldots 2.1$ permutations, and there would then be $\alpha \ldots 2.1 \times \beta \ldots 2.1 \times x$ permutations of n things: γ of them still being *like*. And if we suppose γ things *different*, then they are all different, and their permutations would be $n \ldots . 2.1$, and therefore $x = \frac{n \ldots . 2.1}{\alpha \ldots . 2.1 \times \beta \ldots . 2.1 \times \gamma \ldots . 2.1}$.

Ex. How many words may be made with the letters of the expression $a^4b^3c^2$?

Here we have 9 letters, of which 4 are a's, 3 b's, and 2 c's,

$$\therefore x = \frac{9.8.7.6.5.4.3.2.1}{4.3.2.1 \times 3.2.1 \times 2.1} = 180.$$

Cases of this description are termed variations with repetitions.

EXERCISE LXIX.

Find the number of permutations:

1. Of a^2bc.
2. Of $a^3b^5c^2$ and a^2b^4cd.
3. Of $a^2b^2c^2$ and $a^3b^7c^4d$.
4. Of $a^2b^{10}cd^2e^3$ and $a^5b^5c^4d^2$.
5. Of a^n and $a^{n-1}b$.
6. Of $a^{n-2}b^2$ and $a^{n-3}b^3$.
7. Of $a^{n-4}b^4$ and $a^{n-5}b^5$.
8. How many changes may be rung with 8 bells out of 10?
9. How often can 6 persons change their relative positions at table?
10. The number of variations 2 together is to the number 3 together as 1 : 4. Find the number of things.
11. The number of variations 2 together : the number of things : : 10 : 1. Find the number of things.
12. How many different words may be formed of the letters of the words *Ganges*, *Calcutta*, *Celebes*, *Philippine*?
13. Of how many things are the permutations 40320?
14. Sixty different words can be made from 5 letters. Find the number of a's among them.

The *combinations* of a number of quantities are the different collections that can be made of them without regard to their arrangement.

No two combinations can have the same quantities: for instance, a and b admit of only one combination, because ab, ba are composed of the same quantities; but if a third quantity c be added, we can make three combinations of two quantities out of them, because the third quantity c may be added to each of the two former, thus: ab, ac, bc; and if we add a fourth quantity d, we can make six combinations of two quantities out of the four, since the new quantity d may be combined with each of the former ones, thus: ab, ac, bc, ad, bd, cd, and so on.

To find the number of combinations of n things taken r together.

Let nCr be the number of combinations. Now, since each combination of r quantities will admit of r variations,

$$\therefore \ \{nCr\} \times rVr = nVr = n(n-1)(n-2) \ldots \{n-(r-1)\},$$

$$\therefore \ nCr = \frac{n(n-1)(n-2)\ldots\{n-(r-1)\}}{1 \cdot 2 \cdot 3 \ldots\ldots\ldots r}.$$

From this we see that $nC1 = \frac{n}{1}$, $nC2 = \frac{n(n-1)}{1 \cdot 2}$, $nC3 = \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}$, which we shall find are the same as the coefficients in the expansion of a binomial.

Ex. Find the number of combinations of 8 things taken 3 together.

Here $n=8$, $r=3$,

$$\therefore \ nCr = \frac{8 \cdot 7 \cdot 6}{1 \cdot 2 \cdot 3} = 56.$$

The number of combinations of n things taken r together is equal to the number of them taken $(n-r)$ together.

It is plain that the n things are formed into two sets $(n-r)$ and r when r things are taken from them to form a combination. And if $(n-r)$ things are formed into a combination, this may be called complementary to that of r. And since each combination of r things has its complementary combination of $(n-r)$ things, therefore the number of combinations of n things taken r together is equal to the number of them taken $(n-r)$ together.

EXERCISE LXX.

1. How many combinations can be made out of a, b, c, d, e, f?

2. How many different yoke of oxen may be selected from 18 oxen?

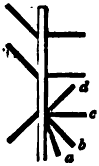
3. There are 4 companies of soldiers, in each of which there are 9 men. In how many ways can 9 men be chosen, 1 from each company?

4. Into how many different triangles may an octagon be divided by drawing lines from the angular points?

5. How often may a different party of 6 be formed from 10 persons?

6. How often may a different guard of 4 men be posted out of 20?

7. A person has 4 coats, 7 waistcoats, and 5 pairs of trousers. In how many different suits may he appear?



8. How many different throws may be made with 8 dice?

9. This cut is a representation of a telegraph with 6 indicators, each of which may be made to take 4 different positions, vertical, slanting downwards, horizontal, and slanting upwards, as *a*, *b*, *c*, *d*. How many different figures may the telegraph be made to present?

CHAPTER XVII.

THE BINOMIAL THEOREM FOR POSITIVE WHOLE EXPONENTS.

The Binomial Theorem is a formula discovered by Sir Isaac Newton, by the help of which the mechanical and often tedious process of multiplication may be obviated, and the expansion of a binomial with any exponent, positive, negative, integral, or fractional, be written down at once.

By the article on Involution we know that

$$(x+a)(x+b)=x^2+\left.\begin{array}{l}a\\+b\end{array}\right\}x+ab$$

$$(x+a)(x+b)(x+c)=x^3+\left.\begin{array}{l}a\\+b\\+c\end{array}\right\}x^2+\left.\begin{array}{l}ab\\+ac\\+bc\end{array}\right\}x+abc$$

$$(x+a)(x+b)(x+c)(x+d)=x^4+\left.\begin{array}{l}a\\+b\\+c\\+d\end{array}\right\}x^3+\left.\begin{array}{l}ab\\+ac\\+ad\\+bc\\+bd\\+cd\end{array}\right\}x^2+\left.\begin{array}{l}abc\\+abd\\+acd\\+bcd\end{array}\right\}x+abcd$$

And on a nearer examination of these products we perceive the general law of involution, which may be expressed by the following formula, if we denote the 1st, 2d, 3d, 4th .... α class of combination, with repetitions for α things $a, b, c \ldots m$ by the letters $A, B, C, D \ldots M$

$$(x+a)(x+b)(x+c)\ldots\ldots(x+m)=x^\alpha+Ax^{\alpha-1}+Bx^{\alpha-2}+Cx^{\alpha-3}+\ldots\ldots+Lx+M.$$

To expand $(x+a)^n$, and to arrange the expansion according to the descending powers of x.

By the preceding formula,

$$(x+a)^n=x^n+Ax^{n-1}+Bx^{n-2}+Cx^{n-3}+\ldots\ldots+Mx+N,$$

in which the coefficients $A, B, C \ldots M, N$ are to be determined.

For this purpose, let us consider any term Kx^{n-k} of the preceding series. K is the k^{th} class of combinations without repetitions for n things, because

n factors have to be multiplied together. Each combination of this class is, in the present case, where all the things are equal to one another, reduced to a^k, and we must now multiply a^k by the number of combinations of the k^{th} class, in order to get all the combinations of the k^{th} class.

The number of combinations of k^{th} class is $\frac{n(n-1)\ldots.(n-k+1)}{1\,.\,2\,.\,3\ldots\ldots k}$.

So that $K=\frac{n(n-1)\ldots.(n-k+1)}{1\,.\,2\,.\,3\ldots\ldots k}a^k$, and if we substitute for k any number from 1 to n, we obtain

$$\begin{aligned}(x+a)^n=x^n+\frac{n}{1}x^{n-1}a&+\frac{n(n-1)}{1\,.\,2}x^{n-2}a^2\\&+\frac{n(n-1)(n-2)}{1\,.\,2\,.\,3}x^{n-3}a^3\\&+\frac{n(n-1)(n-2)(n-3)}{1\,.\,2\,.\,3\,.\,4}x^{n-4}a^4\\&+\ldots\ldots\ldots\ldots\ldots\\&+\frac{n(n-1)\ldots3\,.\,2}{1\,.\,2\,.\,3\,..\,(n-1)}xa^{n-1}\\&+\frac{n(n-1)\ldots.2\,.\,1}{1\,.\,2\,.\,3\ldots.n}a^n,\end{aligned}$$

which may be thus written:

$$(x+a)^n=x^n+\frac{n}{1}x^{n-1}a+\frac{n(n-1)}{1\,.\,2}x^{n-2}a^2+\frac{n(n-1)(n-2)}{1\,.\,2\,.\,3}x^{n-3}a^3+\ldots$$

the series advancing according to the descending powers of x, until the coefficients of x disappear, when the expansion is completed.

If the first and last coefficients are compared, we easily find, by throwing out similar factors in numerator and denominator, that

$$\begin{aligned}\frac{n(n-1)\ldots.2\,.\,1}{1.2.3\ldots\ldots n}&=1,\\\frac{n(n-1)\ldots.3\,.\,2}{1.2.3\ldots.(n-1)}&=\frac{n}{1},\\\frac{n(n-1)\ldots.4\,.\,3}{1.2.3\ldots.(n-2)}&=\frac{n(n-1)}{1\,.\,2},\\\frac{n(n-1)\ldots.5\,.\,4}{1.2.3\ldots.(n-3)}&=\frac{n(n-1)(n-2)}{1\,.\,2\ldots3},\\\&c.&=\&c.\end{aligned}$$

We may also observe that the coefficients of the terms equally distant from the beginning and end of the series are the same.

If we write in the above binomial a for x, and b for a, the expansion becomes

$$\begin{aligned}(a+b)^n=a^n+\frac{n}{1}a^{n-1}b&+\frac{n(n-1)}{1\,.\,2}a^{n-2}b^2\\&+\frac{n(n-1)(n-2)}{1\,.\,2\,.\,3}a^{n-3}b^3\\&+\ \&c.\end{aligned}$$

b may also be negative. If we write $-b$ for b, then

$$\begin{aligned}(a-b)^n=a^n-\frac{n}{1}a^{n-1}b+\frac{n(n-1)}{1\,.\,2}a^{n-2}b^2&\\ -\frac{n(n-1)(n-2)}{1\,.\,2\,.\,3}a^{n-3}b^3&\\ +\ \&c.&\end{aligned}$$

Bringing the two formulæ together,

$$\begin{aligned}(a\pm b)^n=a^n\pm\frac{n}{1}a^{n-1}b+\frac{n(n-1)}{1\,.\,2}a^{n-2}b^2&\\ \pm\frac{n(n-1)(n-2)}{1\,.\,2\,.\,3}a^{n-3}b^3&\\ +\ \&c.&\end{aligned}$$

We see in the expansion of $(a-b)$ that those terms of the series are negative in which the exponent of b is *odd*.

If we denote the coefficients of each term of the series, beginning at the second, by the letters $A, B, C, D, E \ldots.$

$$\begin{aligned}\text{Then }A&=\frac{n}{1},\\ B&=\frac{n(n-1)}{1\,.\,2}=\frac{n}{1}\,.\,\frac{n-1}{2}=\frac{n-1}{2}A,\\ C&=\frac{n(n-1)(n-2)}{1\,.\,2\,.\,3}=\frac{n(n-1)}{1\,.\,2}\,.\,\frac{n-2}{3}=\frac{n-2}{3}B,\\ D&=\frac{n(n-1)(n-2)(n-3)}{1\,.\,2\,.\,3\,.\,4}=\frac{n(n-1)(n-2)}{1\,.\,2\,.\,3}\,.\,\frac{n-3}{4}=\frac{n-3}{4}C.\end{aligned}$$

Employing these expressions, the expansion of $(a\pm b)^n$ becomes

$$\begin{aligned}(a\pm b)^n=a^n\pm Aa^{n-1}b+\frac{n-1}{2}Aa^{n-2}b^2&\\ \pm\frac{n-2}{3}Ba^{n-3}b^3&\\ +\frac{n-3}{4}Ca^{n-4}b^4&\\ \pm\ \&c.&\end{aligned}$$

This form of the series is especially convenient for numerical calculations.

For example, to expand $(a\pm b)^{11}$. Here $n=11$,

$$\begin{aligned}\text{and }A&=\tfrac{11}{1}=11,\\ B&=\tfrac{10}{2}A=\tfrac{10}{2}\,.\,11=55,\\ C&=\tfrac{9}{3}B=\tfrac{9}{3}\,.\,55=165,\\ D&=\tfrac{8}{4}C=\tfrac{8}{4}.165=330,\\ E&=\tfrac{7}{5}D=\tfrac{7}{5}.330=462,\\ F&=\tfrac{6}{6}E=\tfrac{6}{6}.462=462.\end{aligned}$$

From which point the coefficients will be written in a reverse order, so that

$$\begin{aligned}(a\pm b)^{11}=a^{11}\pm 11a^{10}b+55a^9b^2\pm 165a^8b^3+330a^7b^4\pm 462a^6b^5+&\\ 462a^5b^6\pm 330a^4b^7+165a^3b^8\pm 55a^2b^9+11ab^{10}\pm b^{11}.&\end{aligned}$$

We may also by this Theorem expand a quantity consisting of more than two terms, thus:

$$(a+b+c)^5=(a+b)^5+5(a+b)^4c+10(a+b)^3c^2+10(a+b)^2c^3+5(a+b)c^4+c^5$$

in which each binomial is yet to be expanded. And similarly for any number of terms.

A table of the coefficients of binomials may easily be constructed, thus:

$$\text{Let } (1+x)^n=1+Ax+Bx^2+Cx^3+\ .\ .\ .\ .\ .\ +Mx^n,$$
$$(1+x)^{n+1}=1+A'x+B'x^2+C'x^3+\ .\ .\ .\ .\ +M'x^n+Nx^{n+1},$$

and if the first series be multiplied by $1+x$, we shall obtain

$$\left.\begin{array}{r}1+A\\+1\end{array}\right\}x+\left.\begin{array}{r}B\\+A\end{array}\right\}x^2+\left.\begin{array}{r}C\\+B\end{array}\right\}x^3+\ .\ .\ .\ .\ \left.\begin{array}{r}+M\\+L\end{array}\right\}x^n+Mx^{n+1}$$
$$=1+A'x+B'x^2+C'x^3+\ .\ .\ .\ .\ .\ M'x^n+N'x^{n+1}.$$

Consequently,

$$\begin{array}{ll}1=1, & C'=B+C,\\ A'=1+A, & .\ .\ .\ .\ .\ .\\ B'=A+B, & M'=L+M,\\ & N'=M.\end{array}$$

From which it appears that from the binomial coefficient of every power, the coefficient of the next higher power may be found by addition, and that thus a table of binomial coefficients may be constructed.

We give below the beginning of such a table, in which are shown the values of the quantities $\frac{n(n-1)(n-2)\ .\ .\ .\ .\ (n-k+1)}{1\ .\ 2\ .\ 3\ .\ .\ .\ .\ .\ .\ .\ k}$, for all the positive values of n from 1 to 15, and for all positive values of k from 1 to 15. In the first horizontal row are the values of n, and in the first vertical row are the values of k. If we conceive a vertical line to be drawn through a value of n, and a horizontal line drawn through a value of k, we shall find at the point of section of these two lines the corresponding value of

$$\frac{n(n-1)(n-2)\ .\ .\ .\ .\ (n-k+1)}{1\ .\ 2\ .\ 3\ .\ 4\ .\ .\ .\ .\ k}.$$

The values of the expression above when $n=9$ we obtain from the column in the table having $n=8$, in the following manner:

$$\begin{array}{ccc}1+8=9 & 56+70=126 & 28+8=36\\ 8+28=36 & 70+56=126 & 8+1=9\\ 28+56=84 & 56+28=84 & 1+0=1.\end{array}$$

EXERCISE LXXI.

Expand:

1. $(a\pm b)^6$.
2. $(a\pm b)^9$.
3. $(1\pm x)^{11}$.
4. $(5-4x)^4$.
5. $(3-2x^2)^6$.
6. $(\frac{1}{2}x+2y)^7$.
7. $(a^3+3ab)^9$.
8. $(3ac-2bd)^5$.
9. $(5a^2c^2d-4abd^2)^4$.
10. $\left(\frac{2ac}{b^2}+\frac{1}{4}bc^2d\right)^6$.
11. $(\sqrt{a}\pm\sqrt{b})^4$.
12. $(\sqrt{a}\pm\sqrt{b})^7$.
13. $(a+b)^n+(a-b)^n$.
14. $(a+b)^n-(a-b)^n$.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
2		1	3	6	10	15	21	28	36	45	55	66	78	91	105
3			1	4	10	20	35	56	84	120	165	220	286	364	455
4				1	5	15	35	70	126	210	330	495	715	1001	1365
5					1	6	21	56	126	252	462	792	1287	2002	3003
6						1	7	28	84	210	462	924	1716	3003	5005
7							1	8	36	120	330	792	1716	3432	6435
8								1	9	45	165	495	1287	3003	6435
9									1	10	55	220	715	2002	5005
10										1	11	66	286	1001	3003
11											1	12	78	364	1365
12												1	13	91	455
13													1	14	105
14														1	15
15															1

EXERCISE LXXII.

1. $(a \pm b\sqrt{-1})^n$.
2. $(a+b\sqrt{-1})^n+(a-b\sqrt{-1})^n$.
3. $\frac{(a+b\sqrt{-1})^n-(a-b\sqrt{-1})^n}{\sqrt{-1}}$.
4. What is the third term of $(a+b)^{15}$?
5. What is the fifth term of $(a+b)^{16}$?
6. What is the sixth term of $(a-b)^{30}$?
7. Required the fourth term of $(a-b)^{100}$.
8. Required the fifth term of $(a^2-b^2)^{12}$.
9. Required the ninth term of $(2ab-cd)^{14}$.
10. What is the middle term of the series $(a-b)^{16}$?
11. What is the middle term of the series $(a-b)^{18}$?
12. Required the two middle terms of $(a-b)^{17}$.
13. Required the two middle terms of $(a-b)^{19}$.

CHAPTER XVIII.

THE GENERAL BINOMIAL THEOREM.

IT will be seen that the preceding remarks, made in reference to binomials with positive whole exponents, hold equally for those with *negative* and *fractional* indices, except that in these cases the series will not terminate. To prove the Theorem for fractional and negative indices is the object of the following remarks. But before we proceed it will be necessary to investigate the following proposition.

If $a+bx+cx^2+dx^3+\dots=A+Bx+Cx^2+Dx^3+\dots$ for every possible value of x, then a shall equal A, $b=B$, $c=C$, &c.; that is, the coefficients of the like powers of x shall be equal.

Since x may be of any value, let $x=0$.

Then $a=A$, and expunging these from the equation,

we have $bx+cx^2+dx^3+\dots=Bx+Cx^2+Dx^3+\dots$

and dividing by x,

$$b+cx+dx^2+\dots=B+Cx+Dx^2+\dots$$

$\therefore$ $b=B$, and again expunging these from the equation, we shall find that $c=C$, and $d=D$, &c.

This is termed the method of Indeterminate Coefficients, a theorem exceedingly useful in many algebraical operations, and which we shall presently employ.

And first, let $n=\frac{1}{2}$ in the binomial $(1+x)^n$.

$(1+x)^{\frac{1}{2}}=\sqrt{1+x}$, and this will be expanded into a series by the method just alluded to, if we put

$$\sqrt{1+x}=A+Bx+Cx^2+Dx^3+Ex^4+\dots$$

Squaring, we get

$$1+x=AA+\left.\begin{array}{r}AB\\+BA\end{array}\right\}x+\left.\begin{array}{r}AC\\+BB\\+CA\end{array}\right\}x^2+\left.\begin{array}{r}AD\\+BC\\+CB\\+DA\end{array}\right\}x^3+\left.\begin{array}{r}AE\\+BD\\+CC\\+DB\\+EA\end{array}\right\}x^4+\dots$$

or $$1+x=AA+2ABx+\left.\begin{array}{r}2AC\\+BB\end{array}\right\}x^2+\left.\begin{array}{r}2AD\\+2BC\end{array}\right\}x^3+\left.\begin{array}{r}2AE\\+2BD\\+CC\end{array}\right\}x^4+\dots$$

in which we must now so determine the coefficients A, B, C, D, E that the equation may be satisfied for every value of x. This will be effected if we so determine them that they satisfy the following equations:

$$AA=1,$$
$$2AB=1,$$
$$2AC+BB=0,$$

$$AD+BC=0,$$
$$2AE+2BD+CC=0,$$
$$AF+BE+CD=0,$$

from which equations we can now determine A, B, C, D, E, &c.

$$A=1,\ B=\tfrac{1}{2},\ C=-\tfrac{1}{8},\ D=\tfrac{1}{16},\ E=-\tfrac{5}{128},\ \text{\&c.}$$

Therefore $(1+x)^{\frac{1}{2}}=1+\frac{1}{2}x-\frac{1}{8}x^2+\frac{1}{16}x^3-\frac{5}{128}x^4+\ \ldots.$

or $(1+x)^{\frac{1}{2}}=1+\frac{\frac{1}{2}}{1}x+\frac{\frac{1}{2}(\frac{1}{2}-1)}{1\,.\,2}x^2+\frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{1\,.\,2\,.\,3}x^3+\frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)(\frac{1}{2}-3)}{1\,.\,2\,.\,3\,.\,4}x^4+\ \ldots$

a series which clearly follows the general law of binomials.

In an exactly similar manner may we expand into series $(1+x)^{\frac{1}{3}}$, $(1+x)^{\frac{1}{4}}$, $(1+x)^{\frac{1}{5}}$. So also the method of Indeterminate Coefficients may be applied to the expansion of binomials with negative exponents. But for the *general* expansion of $(1+x)^n$, where n may be *positive*, *negative*, *integral*, or *fractional*, many artifices must be employed, as will appear in the following paragraphs.

By the method of Indeterminate Coefficients:

Let $(1+x)^n=A+Bx+Cx^2+Dx^3+Ex^4+Fx^5+\ \ldots.$ where the coefficients do not depend upon x, but upon the exponent n; and let us attempt to form equations from which we may determine these coefficients. To this end, put $x+y$ for x, then will

$$\{1+(x+y)\}^n=A+B(x+y)+C(x+y)^2+D(x+y)^3+\ \ldots.$$

And if we expand the powers of $x+y$ according to the theorem for positive integral exponents, and arrange the series according to the powers of y, but for brevity employing that term of it containing the simple power of y,

$$\begin{aligned}(1+x+y)^n=&A+Bx+Cx^2+Dx^3+Ex^4+\ \ldots.\\ &+\{B+2Cx+3Dx^2+4Ex^3+5Fx^4+\ \ldots\}y\\ &+\ \ \text{\&c.}\\ =&(1+x)^n\\ &+\{B+2Cx+3Dx^2+4Ex^3+5Fx^4+\ \ldots.\}y\quad(\gamma)\\ &+\ \ \text{\&c.}\end{aligned}$$

But $1+x+y=(1+x)\left\{1+\frac{y}{1+x}\right\}$

$$\therefore\ \ (1+x+y)^n=(1+x)^n\left\{1+\frac{y}{1+x}\right\}^n.$$

And if we put $\frac{y}{1+x}$ for x in the original equation, then

$$\left\{1+\frac{y}{1+x}\right\}^n=A+B\left(\frac{y}{1+x}\right)+C\left(\frac{y}{1+x}\right)^2+D\left(\frac{y}{1+x}\right)^3+\ \ldots.$$

$$\begin{aligned}(1+x)^n\left\{1+\frac{y}{1+x}\right\}^n=&A(1+x)^n+B(1+x)^{n-1}y+C(1+x)^{n-2}y^2\\ &+D(1+x)^{n-3}y^3+\ \ldots\end{aligned}$$

$$\therefore\ \ (1+x+y)^n=A(1+x)^n+B(1+x)^{n-1}y+\ \ldots.\ (\delta)$$

Therefore for every value of n, positive, negative, fractional, or integral, $B=n$. Consequently, according to the preceding,

$$\begin{aligned}B&=\frac{n}{1},\\ C&=\frac{n(n-1)}{1\,.\,2},\\ D&=\frac{n(n-1)(n-2)}{1\,.\,2\,.\,3},\\ E&=\frac{n(n-1)(n-2)(n-3)}{1\,.\,2\,.\,3\,.\,4},\\ \&c. &= \&c.\end{aligned}$$

Also, $(1+x)^n=1+\frac{n}{1}x+\frac{n(n-1)}{1\,.\,2}x^2+\frac{n(n-1)(n-2)}{1\,.\,2\,.\,3}x^3+\,\ldots\,.$

Remark. Since $a\pm x=a\left(1\pm\frac{x}{a}\right)$, $(a\pm x)^n=a^n\left(1\pm\frac{x}{a}\right)^n$,

so $(a\pm x)^n=a^n\{1\pm\frac{n}{1}\left(\frac{x}{a}\right)+\frac{n(n-1)}{1\,.\,2}\left(\frac{x}{a}\right)^2\pm\frac{n(n-1)(n-2)}{1\,.\,2\,.\,3}\left(\frac{x}{a}\right)^3+\,\ldots\,\}$

i. e. $(a\pm x)^n=a^n\pm\frac{n}{1}a^{n-1}x+\frac{n(n-1)}{1\,.\,2}a^{n-2}x^2\pm\frac{n(n-1)(n-2)}{1\,.\,2\,.\,3}a^{n-3}x^3+\,\ldots\,.$

and therefore the Theorem is true for every kind of exponent.

Remark. $(a\pm x)^{-n}=a^{-n}\mp\frac{n}{1}a^{-n-1}x+\frac{n(n+1)}{1\,.\,2}a^{-n-2}x^2$

$$\mp\frac{n(n+1)(n+2)}{1\,.\,2\,.\,3}a^{-n-3}x^3+\,\ldots\,.$$

$$(a\pm x)^{\frac{m}{n}}=a^{\frac{m}{n}}\pm\frac{m}{n}a^{\frac{m}{n}-1}x+\frac{m(m-n)}{n\,.\,2n}a^{\frac{m}{n}-2}x^2\pm\frac{m(m-n)(m-2n)}{n\,.\,2n\,.\,3n}a^{\frac{m}{n}-3}x^3+\,\ldots\,.$$

By means of the Theorem just investigated roots of every kind may be approximatively extracted from given quantities. But the calculation must always be so arranged that the series which the Binomial Theorem gives for the root sought, *converge;* that is, that the quantities which are gradually obtained by taking the first member of the series, and the two, three, four, five, or six of its first successive terms, gradually approach nearer to that quantity which is to be approximatively obtained by help of the series, and be brought as near to that quantity as can possibly be desired; so that the defect, absolutely considered, may be made smaller than any positive quantity, be that small as it may, if only a sufficient number of terms be taken into the series in uninterrupted succession from the commencement. The more quickly the quantities in question approach the quantity to be calculated, the more quickly, it is said, does the series *converge.*

A few examples will explain the preceding remarks.

To find an approximate value of $\sqrt{2}$.

$$\begin{aligned}\sqrt{2}&=\tfrac{1}{5}\sqrt{50}=\tfrac{1}{5}(49+1)^{\frac{1}{2}}=\tfrac{7}{5}(1+\tfrac{1}{49})^{\frac{1}{2}}\\ &=\tfrac{7}{5}\{1+\tfrac{1}{2}(\tfrac{1}{49})-\frac{1}{2\,.\,4}(\tfrac{1}{49})^2+\frac{1\,.\,3}{2\,.\,4\,.\,6}(\tfrac{1}{49})^3-\frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6\,.\,8}(\tfrac{1}{49})^4+\,\ldots\,\}.\end{aligned}$$

This series converges very quickly, and by means of it we can easily find the root accurately up to ten places of decimals, if we change the single terms into decimal fractions. To find an approximate value of $\sqrt[n]{N}$ we must either put $N=a^n+b$ or $N=a^n-b$, so that a^n comes as near as possible to N, and also b is, in relation to a^n, as small as possible.

$$\sqrt[n]{N}=\sqrt[n]{(a^n\pm b)}=a\sqrt[n]{\left(1\pm\frac{b}{a^n}\right)}$$

$$=a\left\{1\pm\frac{1}{n}\left(\frac{b}{a^n}\right)-\frac{n-1}{n\,.\,2n}\left(\frac{b}{a^n}\right)^2\pm\frac{(n-1)(2n-1)}{n\,.\,2n\,.\,3n}\left(\frac{b}{a^n}\right)^3\right.$$

$$\left.-\frac{(n-1)(2n-1)(3n-1)}{n\,.\,2n\,.\,3n\,.\,4n}\left(\frac{b}{a^n}\right)^4\pm\ldots\ldots\right\}$$

And the root is thus taken with a great degree of accuracy.

EXERCISE LXXIII.

1. $(a+b)^{\frac{7}{2}}$
2. $(x-z)^{\frac{2}{3}}$
3. $(1\pm z)^{\frac{1}{3}}$
4. $(1\pm z)^{\frac{1}{4}}$
5. $(1-x)^{-4}$
6. $(1+x)^{-2}$
7. $(1-x)^{-3}$
8. $(1\pm z)^{\frac{1}{3}}$
9. $(x+a)^{-\frac{1}{4}}$
10. $(3a-5b)^{-\frac{2}{3}}$
11. $(m-n)^{-\frac{3}{4}}$
12. $(x^3+y^3)^{-\frac{1}{3}}$
13. $\left(\frac{a^{10}}{a^5-x^5}\right)^{\frac{2}{3}}$
14. $\left(\frac{a-2b}{21x^2}\right)^{-\frac{2}{3}}$
15. Find the first four terms of the sixth negative power of $x^{2m}+y^{2n}$.
16. Find the first four terms of the tenth negative power of x^3+xy^2.
17. Find an approximate value of $\sqrt[3]{7}$.
18. Find the square root of 5.
19. Find an approximate value of square root of 50.
20. Find the approximate value of the p^{th} root of (N^p+x).

CHAPTER XIX.

THE THEORY OF DECIMAL FRACTIONS.

A DECIMAL fraction is one whose numerator is any whole number whatever, but whose denominator is a power of 10.

According to this definition a decimal fraction will be symbolically represented by $\frac{A}{10^n}$, where n is the number of decimal places, and A the integral number which the decimal would represent if we removed the point. Thus, $2{\cdot}59=\frac{259}{10^2}$; A in this case denoting 259, and n equalling 2.

To prove the rule for multiplication.

Let $\frac{A}{10^m}$ and $\frac{B}{10^n}$ be the decimals to be multiplied,

$$\text{then}\quad \frac{A}{10^m}\times\frac{B}{10^n}=\frac{AB}{10^{m+n}};$$

AB representing the product of the decimals considered as whole numbers, and $m+n$ the number of decimal places in the multiplier and the multiplicand together. And hence the rule in arithmetic: Multiply as in whole numbers, and point off in the product as many figures as there are decimal places in the multiplier and multiplicand together.

To prove the rule for division of decimals.

Let $\frac{A}{10^m}$ and $\frac{B}{10^n}$ be the decimals,

$$\text{then}\quad \frac{A}{10^m} \div \frac{B}{10^n} = \frac{A}{10^m} \times \frac{10^n}{B} = \frac{A}{B} \cdot \frac{10^n}{10^m}.$$

Therefore if $m > n$, the quotient will be $\frac{A}{B} \cdot \frac{1}{10^{m-n}}$, *i. e.* it will contain as many decimal places as the dividend has more than the divisor.

If $m=n$, the quotient will have no decimal places.

If $m < n$, the quotient will be $\frac{A}{B} \cdot 10^{n-m}$, *i. e.* the quotient must be multiplied by that power of 10 denoted by the excess of n over m.

To prove the rule for changing a vulgar into a decimal fraction.

Let $\frac{A}{B}$ be the vulgar fraction, then $\frac{A \times 10^m + B}{10^m}$ will represent the decimal fraction, and obviously the rule in arithmetic: Add ciphers at pleasure to the numerator, and divide by the denominator. The quotient will have as many decimal places as we add ciphers.

To prove the rule for reducing a circulating decimal to a vulgar fraction.

Let $\cdot BCCC$ represent a circulating decimal, in which B is the non-recurring, and C the recurring part, and let B contain m digits, and C n digits.

$$\text{Let}\quad P = \cdot BCCC, \text{ then } 10^m P = B \cdot CCC \ldots$$

$$\text{and}\quad 10^{m+n} P = BC \cdot CCC \ldots.$$

$$\therefore\quad (10^{m+n} - 10^m)P = BC - B,\ \therefore\ P = \frac{BC-B}{10^{m+n}-10^m} = \frac{BC-B}{10^m(10^n-1)},$$

a formula in agreement with the rule: To as many 9's as there are figures in the repetend, annex as many ciphers as there are figures in the non-recurring part for the denominator. Subtract the finite part from that portion of the decimal extending to the end of the first period for the numerator.

To find whether a given vulgar fraction is equal to a finite or infinite decimal.

It is manifest that if the denominator of the proposed fraction be 10, or any power of 10, it will reduce to a finite decimal; and from this it follows that if the denominator can be resolved into factors, one of which is a power of 2, and the other of 5, or if we can multiply the numerator and denominator by any number, so as to make the denominator of this form, the fraction will reduce to a finite decimal. Under no other circumstances will it terminate.

Let $\frac{A}{B}$ be a fraction in its lowest terms; then, if B is a power of 10, or, which is the same thing, if it is resolvable into the form $2^m \cdot 5^n$, the decimal will terminate, and with the places denoted by the greater of the powers m, n.

For if $m > n$, then $\frac{A}{2^m . 5^n}$ being multiplied so as to render the denominator $10^m = \frac{A . 5^{m-n}}{2^m . 5^m} = \frac{A . 5^{m-n}}{10^m}$, in which m denotes the number of decimal places.

If $n > m$, then $\frac{A}{2^m . 5^n} = \frac{A . 2^{n-m}}{2^n . 5^n} = \frac{A . 2^{n-m}}{10^n}$, where n denotes the number of places.

CHAPTER XX.

INTEREST.

Let $r=$ rate per cent of 1*l.* for 1 year.
$t=$ months, weeks, or days in 1 year.
$A=$ principal.
$n=$ months, weeks, or days for which principal is lent.
$B=$ amount of principal for time n.

Then $nr=$ the interest of 1*l.* for time n.

$Anr=$ the interest of A*l.* for time n, or the interest = principal × time × interest of 1*l.* for 1 year.

The amount $=A+Anr=A(1+nr)$.

If i be the interest for 100*l.* for 1 year, then $r=\frac{i}{100}$.

And (i) $\frac{Ani}{100t}=$ interest of principal A for time n at c per cent per annum. (ii) The amount $B=A+\frac{Ani}{100t}$, whence we have also (iii) $A=\frac{100Bt}{100t+ni}$, (iv) $i=\frac{100t(B-A)}{An}$ and (v) $n=\frac{100t(B-A)}{Ai}$, the use of which equations will appear by the following examples.

Ex. 1. What is the amount of 550*l.* at 4 per cent in 7 months?

Here $A=550$, $i=4$, $t=12$, and $n=7$.

$$\therefore \text{(ii) } B=A+\frac{Ani}{100t}=550+\frac{550 \times 7 \times 4}{100 \times 12}=562\tfrac{5}{6}l.$$

Ex. 2. What is the interest of 1*l.* for 1 day at 5 per cent per annum?

Here $A=1$, $n=1$, $i=5$, $t=365$.

$$\therefore \text{(i) } \frac{Ani}{100t}=\frac{5}{100 \times 365}=\cdot 0001369863l.$$

Ex. 3. What sum in ready money is equivalent to 600*l.* due 9 months hence, allowing 5 per cent discount?

Here $i=5$, $t=12$, $n=9$, and $B=600$.

$$\therefore \text{(iii) } \frac{100Bt}{100t+ni}=\frac{100 \times 600 \times 12}{100 \times 12+9 \times 5}=578\cdot 313l.=578l.\ 6s.\ 3\tfrac{1}{4}d.$$

Ex. 4. At what rate of interest will 300*l.* in 15 months amount to 330*l.*?

$$\text{By (iv) } i = \frac{100 \times 12 \times 30}{300 \times 15} = 8l. \text{ the rate per cent.}$$

Ex. 5. In how many days will 365*l.* at the rate of 4 per cent amount to 400*l.*?

$$\text{By (v) } n = \frac{100t(B-A)}{Ai} = \frac{100 \times 365 \times 35}{365 \times 4} = 875 \text{ days, the time required.}$$

COMPOUND INTEREST.

Let $R =$ the amount of 1*l.* for 1 year $= 1 + r$.
$P =$ the sum put out at interest.
$n =$ the number of years it is lent for.
$a =$ its amount in that time.
$A =$ an annuity forborne n years.
$m =$ its amount in that time.
$v =$ the present value of the annuity for the same time.

Then since 1*l.* placed at interest in the first year is increased to R, $1 : R : : R : R^2$ is the amount of 1*l.* in 2 years, and $1 : R : : R^2 : R^3$ the amount in 3 years; whence it appears that R^n, or R raised to the power whose exponent is the number of years, will be the amount of 1*l.* in those years.

But 1*l.* : R^n (its amount) : : $P : a$, ∴ (i) $PR^n = a$. And because the amount of 1*l.* in n years $= R^n$, its increase in that time will be $R^n - 1$; but its interest for 1 year $= R - 1$,

$$\therefore R - 1 : R^n - 1 : : A : m, \quad \therefore \text{(ii) } m = \frac{A(R^n - 1)}{R - 1}.$$

And further, since 1*l.* ready money is equal to R^n to be received after n years, $\therefore R^n : 1 : : \frac{A(R^n - 1)}{R - 1} : v$,

$$\text{and } \therefore \text{(iii) } v = \frac{A\left(1 - \frac{1}{R^n}\right)}{R - 1}.$$

From which equations, and others derivable from them, may be solved the various questions relating to compound interest, annuities in arrear, and present values of annuities.

Thus, $PR^n = a$, $\therefore P = \frac{a}{R^n}$, and $R = \left(\frac{a}{P}\right)^{\frac{1}{n}}$, or, for simplicity, employing logarithms,

$$\text{(i) } \log. a = \log. P + n \log. R,$$

$$\text{(ii) } \log. P = \log. a - n \log. R,$$

$$\text{(iii) } \log. R = \frac{\log. a - \log. P}{n},$$

$$\text{(iv) } n = \frac{\log. a - \log. P}{\log. R},$$

which equations serve for all the cases of compound interest.

Again, since $m=\frac{A(R^n-1)}{R-1}$, we have

$$\text{(i)}\quad \log. m=\log. A+\log. (R^n-1)-\log. (R-1)$$

$$\text{(ii)}\quad \log. A=\log. m-\log. (R^n-1)+\log. (R-1)$$

$$\text{(iii)}\qquad n=\frac{\log. (mR-m+A)-\log. A}{\log. R}$$

$$\text{(iv)}\qquad R^n-\frac{mR}{A}+\frac{m}{A}-1=0,$$

equations which relate to annuities in arrear.

Since $v=A\,\dfrac{1-\frac{1}{R^n}}{R-1}$, we obtain

$$\text{(i)}\quad \log. v=\log. A+\log. \left(1-\frac{1}{R^n}\right)-\log. (R-1)$$

$$\text{(ii)}\quad \log. A=\log. v+\log. (R-1)-\log. \left(1-\frac{1}{R^n}\right)$$

$$\text{(iii)}\qquad n=\frac{\log. A-\log. (A+v-vR)}{\log. R}$$

$$\text{(iv)}\quad R^{n+1}-\left(\frac{A}{v}+1\right)R^n+\frac{A}{v}=0,$$

equations which respect the present values of annuities.

Examples.

Ex. 1. To find the amount of 575*l.* in 7 years at 4 per cent per annum compound interest.

By equation (i) $a=756\cdot66l.=756l.\ 13s.\ 2\tfrac{1}{4}d.$

Ex. 2. What principal put to interest will raise a stock of 1000*l.* in 15 years at 5 per cent?

By equation (ii) $P=481\cdot02l.=481l.\ 0s.\ 4\tfrac{3}{4}d.$

Ex. 3. In how long time will 575*l.* amount to 756*l.* 13*s.* 2¼*d.* at 4 per cent?

By equation (iv) $n=7$, the number of years required.

Ex. 4. To find at what rate of interest 481*l.* in 15 years will amount to 1000*l.*

By equation (iii) $R=1\cdot05$ per cent.

Examples might be given to illustrate the remaining equations, but the student will supply them for himself from sources easily within his reach.

CHAPTER XXI.

NOTATION.

NOTATION is the expression of numbers by figures in a common ratio.

Numbers are arranged into *systems* or *scales* according to the common ratio. Thus, a system in which 10 is the common ratio is termed a decimal system of numbers,—such is the one in which all our arithmetical calculations are made. For example, 25926 signifies $6+20+900+5000+20000=6+2.10+9.10^2+5.10^3+2.10^4$. So also if the common ratio, or radix as it is called, is 20, or the system vigesimal, 25645 would be represented by 3425, because $5+2.20+4.20^2+3.20^3=25645$.

From this it is manifest that any number in one system may have its representation in another. Generally, if N be the number, and r the common ratio, then N may assume the form $a''''r^n+a'''r^{n-1}+a''r^{n-2}+\ldots+a'r+a$, in which the coefficients of the powers of r are $< r$. These coefficients may be 0, but can never be $> r$, and therefore they may be of any value from 0 to $r-1$. Hence, in the ternary system these values may be 0, 1, 2; in the undenary 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10; and so on.

Let us take the example $N=a''''r^n+a'''r^{n-1}+a''r^{n-2}+\ldots\ldots+a'r+a$, and show how we obtain the coefficients of the powers of r.

If we divide N by r we get the quotient

$$a''''r^{n-1}+a'''r^{n-2}+\ldots\ldots+a'+\frac{a}{r},$$

a being the remainder will be the last digit or *unit* figure. And if we divide this quotient by r we obtain as remainder a', the last but one digit, or the figure occupying the tens' place in our notation, and so on for the remaining coefficients if we continue the division until the remainder becomes $< r$.

Ex. 1. Express the common number 34567 in the senary scale.

$$\begin{array}{r l}
6)\underline{34567} & \\
6)\underline{5761} & +1 \\
6)\underline{960} & +1 \\
6)\underline{160} & +0 \\
6)\underline{26} & +4 \\
4 & +2. \quad \textit{Ans. } 424011.
\end{array}$$

Ex. 2. Reduce 39824 in the duodenary scale to the scale of 9.

Let t, e, t', t'' represent 10, 11, 12, 13.

$$\begin{array}{r l}
9)\underline{39824} & \\
9)\underline{50te} & +1 \\
9)\underline{692} & +5 \\
9)\underline{90} & +2 \\
9)\underline{10} & +0 \\
1 & +3. \quad \textit{Ans. } 130251.
\end{array}$$

Observe the process in this division by 9. In the first line we say 9 in 45 (for $3\times12+9=45$) 5 times, 9 in $8=0$, with 8 to carry, which is $8\times12=96$, and $96+2=98$, 9 in $98=t$, with 8 over, which again $=96$, and $96+4=100$, and dividing by 9, the quotient $=e$, with 1 remaining. The number carried is always the radix of the system.

We will add examples of the ordinary rules of arithmetic, for it is evident that the same operations can be performed with numbers of any radix as with those in the ordinary scale of 10, the same care being had to the radix as with the 10 in common numbers.

$r=7$. Add together

$$\begin{array}{r} 256431 \\ 2154 \\ 63253 \\ 4152 \\ 31425 \\ 6142 \\ \hline 433253 \end{array}$$

$r=9$.

$$\begin{array}{r} 8426 \\ 274 \\ 38 \\ 126 \\ 54 \\ 38 \\ 2716 \\ \hline 12806 \end{array}$$

$r=4$. Take 12301213 from 300021303.

$$\begin{array}{r} 300021303 \\ 12301213 \\ \hline 221120030 \end{array}$$

$r=14$. From $89t5t'e$ take $7c6t''49$.

$$\begin{array}{r} 89t5t'e \\ 7e6t''49 \\ \hline t'3582 \end{array}$$

Multiply 345621 by 254, in the scale of 7; and $4t9e$ by 347, in the scale of 12.

$$\begin{array}{r} 345621 \\ 254 \\ \hline 2052414 \\ 2431335 \\ 1024542 \\ \hline 132153264 \end{array} \qquad\qquad \begin{array}{r} 4t9e \\ 347 \\ \hline 2t395 \\ 17738 \\ 12859 \\ \hline 146e455 \end{array}$$

Divide 26453210 by 5402, in the scale of 6.

$$\begin{array}{l} 5402)26453210(3152 \\ \underline{25010} \\ 14432 \\ \underline{5402} \\ 50301 \\ \underline{44214} \\ 20430 \\ \underline{15204} \\ 1222 \end{array}$$

Extract the square root of 45327614, in scale of 8.

$$\begin{array}{rl}
 & 45327614(6074 \\
 & 44 \\ \cline{2-2}
1407) & 13276 \\
 & 12461 \\ \cline{2-2}
14164) & 61514 \\
 & 60720 \\ \cline{2-2}
 & 574
\end{array}$$

Squaring 6074, and adding the remainder,

$$\begin{array}{r}
6074 \\
6074 \\ \hline
30360 \\
52644 \\
44550 \\
574 \\ \hline
45327614
\end{array}$$

Exercise LXXIV.

1. Express 3000 in the scale of 4, 5, 6, 7, 8.

2. Express 25783201 in the nonary scale.

3. Express 290 in the scale of 2, 4, 6.

4. Change 524 and 2034 from the senary to the nonary scale.

5. Change *te*94 and 300 from the duodenary to the quaternary scale

6. Change 340006 from the undenary to the quaternary scale.

7. Add together in the senary scale 2530124, 34503, 21345, 120304. 513, 2132, 1401 ; and *te*4196, 349*t*1*e*76, 39325, 4*et*496345, 39*e*4*t*12, 4694712 in the scale of 12.

8. From 34*t*769501 take 23940*t*78 in the undenary scale.

9. Multiply together 54 and 68 in the duodenary scale, and reduce the product to the denary.

10. Change 1546 and 46 from the nonary to the quaternary, then find their sum, difference, product, and quotient.

11. Divide 210041 by 31 (quinary), 2906418 by 45 (duodenary), and 456835 by 65 (undenary).

12. Extract the square root of 601020 (octenary), 4120400 (senary), and 22302021 (quinary).

Miscellaneous Exercises.

1. Multiply $(b-c)a^3+(b^3-b^2c+bc^2-c^3)a^2+3a$ by $b+c$.

2. Divide $119c^2-200cd+408ce-113cf-39d^2+72de+37df-96ef+20f^2$ by $17c+3d+4f$.

3. Simplify $a-b+\frac{ab}{a-b}-\frac{a^2+b^2}{a+b}-\frac{a^2b-2ab^2+3b^2}{a^2-b^2}$.

4. Multiply $\frac{3a^{-5}b^3c^3f^4}{(a+b)^{-m}(c^2+x^2)}$ by $\frac{a^2b^{-5}c^{-2}f^{-4}}{(a+b)^{m-2}(c^2+x^2)^{-n+4}}$.

5. Solve the equation $\sqrt{1+x\sqrt{x^2+12}}=1+x$.

6. Simplify $\frac{6(a+b)^{-9}}{4(a+b)^{-5}}$ and $\frac{(a+x)^2(a+y)^{-3}}{(a+x)^{-4}(a+y)^{-7}}$

7. Divide the number a into three such parts that the second may be m times and the third n times as great as the first.

8. Rationalise the denominators $\frac{1}{(\sqrt[4]{a}+\sqrt[4]{b})} \cdot \frac{(\sqrt[4]{a}+\sqrt[4]{b})}{(\sqrt[4]{a}-\sqrt[4]{b})}$.

9. In a geometrical proportion there are given the sum of both the middle terms $=a$, the sum of both the outer terms $=b$, and the sum of the biquadrates of all the four terms $=c$. What is the proportion?

10. Reduce to its simplest form $\sqrt[m]{2^m a^{mp+3} b^{mn+5}}+\sqrt[m]{3^m a^{2m-mn+3} b^{m+5}}-\sqrt[m]{a^3b^5c^{2m}}$.

11. Extract the square root of $\frac{7}{4}$, $10\frac{3}{10}$, and ·014.

12. Find m arithmetical means between a and l.

13. Simplify $\dfrac{\sqrt{(1+x)}+\dfrac{1}{\sqrt{(1+x)}}}{1+\dfrac{1}{\sqrt{(1-x^2)}}}$ and $\dfrac{1+\dfrac{\sqrt{(a^2-x^2)}}{\sqrt{(a^2+x^2)}}}{\sqrt{(a^2+x^2)}+\sqrt{(a^2-x^2)}}$.

14. Required the sum of the series 1, $-\frac{1}{2}$, $\frac{1}{4}$, $-\frac{1}{8}$, $\frac{1}{16}$, &c. to infinity.

15. Write down the product of $(x+a)(x-b)(x-c)$ and of $(x+a)(x+b)(x+c)(x+d)$.

16. There is a fraction which becomes $\frac{1}{3}$ if 1 is added to its numerator, and $\frac{1}{5}$ if 3 is added to its denominator. Find it.

17. Find the third term of the expansion of $(x-y)^{\frac{1}{2}}$, and expand $(a+b)$ to the $\frac{3}{4}$ negative power.

18. $5=3\sqrt{x}-5$, $\sqrt{36+x}=18+\sqrt{x}$, $\frac{17-5\sqrt{x}}{11}=-3$.

19. What is the sum of the infinite geometrical series $a-b+\frac{b^2}{a}-\frac{b^3}{a^2}+\frac{b^4}{a^3}-$ &c.?

20. By what vulgar fraction can the periodic decimals ·8686 and ·375375 be expressed? Prove the rule in arithmetic.

21. Reduce to its lowest terms $\frac{x^4-3ax^3-8a^2x^2+18a^3x-8a^4}{a^3-ax^2-8a^2x+6a^3}$ and $\frac{a^4-x^4}{a^3-a^2x-ax^2+x^3}$.

22. Divide the number 90 into four such parts, that if the first be increased by 5, the second decreased by 4, the third multiplied by 3, and the fourth divided by 2, the result in each case shall be exactly the same.

23. How many terms of the series 7, 5, 3, &c. $=-384$, and what is the 18th term of the series 4, -3, &c.?

24. Express in the form of a continued fraction the value of $\frac{1+x}{1-2x+x^2}$ and $\frac{x+1}{x^2-2x+1}$ when $x=\frac{1}{3}$.

25. Extract the square root of $20\frac{1}{4}a^6+49x^6-21ax^5-27a^5x+22\frac{1}{2}a^4x^2+44\frac{1}{4}a^2x^4-72a^3x^3$.

26. Take the cube root of $8x^9-36x^8+125-75x-210x^2+239x^3+42x^7+69x^6-201x^5+66x^4$.

27. Find the G. C. M. of $20y^4+y^2-1$ and $25y^4+5y^3-y-1$, also of $x^4-2a(a-b)x^2+(a^2+b^2)(a-b)x-a^2b^2$ and $x^4-(a-b)x^3+(a-b)b^2x-b^4$.

28. A draper buys cloth for 33*l.* 15*s.*, which he sells again for 48*s.* per piece, gaining as much on the whole as the price of a single piece. At what rate per piece was the cloth bought?

29. Take the roots from the denominators of $\frac{a}{\sqrt{b}}$, $\frac{\sqrt{m}}{\sqrt{n}}$, $\frac{m}{n+\sqrt{p}}$, and $\frac{a}{\sqrt{m}\pm\sqrt{n}}$.

30. Show the equality of $\sqrt{\sqrt{32}+\sqrt{24}}$ and $\sqrt[4]{18}+\sqrt[4]{2}$; of $\sqrt{\sqrt{63}-\sqrt{35}}$ and $\sqrt[4]{43\cdot75}-\sqrt[4]{1\cdot75}$.

31. Divide $\frac{x^{m+3n}y^{7m-8n}}{x^{4m-7n}y^{3m-11n}}$ by $\frac{x^{2m-6n}y^{5m+6n}}{x^{6m-17n}y^{2m+4n}}$.

32. What is the 10th term of the expansion of $(2-5x-7x^2+x^3+3x^4)^5$?

33. Find the permutations of the letters in the word *conformation*.

34. Extract the square root of a^2+x^2, and the cube root of $1-x^3$.

35. How many different combinations, taking four together, can be formed of 4 sets of different things, one containing 10, another 8, the third 6, and the fourth 4?

36. Add together $\frac{1}{3(1+x)}$ and $\frac{2-x}{3(1-x+x^2)}$; $\frac{a}{b^n}$, $\frac{c}{b^{n-r}}$, and $\frac{d}{b^{n-2r}}$.

37. Show that $\frac{1}{(a-b)(a-c)(x+a)}-\frac{1}{(a-b)(b-c)(x+b)}+\frac{1}{(a-c)(b-c)(x+c)}=\frac{1}{(x+a)(x+b)(x+c)}$.

38. The side of a cube is $1\frac{1}{2}$ feet longer than that of another whose solid contents are $2501\frac{7}{8}$ feet less than the former. Compare their solidities.

39. Solve equations (i) $(x+y)^2(x^2+y)=392$ and $(x^2-y)(y^2-x)=88$;

(ii) $(x^2-a^2)^2+2b(x^2-a^2)=c^2$.

40. Required to find two multiples of 13 and 7 whose difference shall be 500.

41. Simplify $\frac{3}{4(1-x)^2}+\frac{3}{8(1-x)}+\frac{1}{8(1+x)}-\frac{1-x}{4(1+x^2)}$.

42. There is a certain number consisting of two places of figures which is equal to 4 times the sum of its digits, and if 27 be added to it the digits will be inverted. What is the number?

43. Divide $ab-a^2+b^2$ by $\sqrt{ab}+\sqrt{(a^2-b^2)}$, and $14-\sqrt{15}-7\sqrt{-3}-2\sqrt{-5}$ by $7-\sqrt{-5}$.

44. Find the square root of $\{x-2\sqrt{(x-1)}\}$ and $\left\{\frac{a^2}{4}+\frac{c}{2}\sqrt{(a^2-c^2)}\right\}$.

45. Extract the cube root of 28·25 and of $\frac{27u^9x^{12}(a^2+x^2)^{-6n}}{8b^{-3n}h^9}$.

46. Show that the square root of 3492 and of this number converted to the septenary scale are equal.

47. Simplify the expressions $\sqrt[m]{\{\sqrt[n]{(\sqrt[n]{a})}\}}$, $\sqrt{(\sqrt[3]{9a^2})}$, $\sqrt[5]{(2b\sqrt[4]{2b})}$ and $\sqrt[4]{\{2a\sqrt[3]{(bc\sqrt{d})}\}}$.

48. Place $\sqrt{(x^2+y^2)}$ under the form $y\left\{1+\left(\frac{x}{y}\right)^2\right\}^{\frac{1}{2}}$.

49. What numbers are as p to q, and the sum of whose squares $=b$?

50. The sum of p arithmetical means between 1 and 19 : the sum of the first $p-2$ of them : : 5 : 3 ; find the number of them.

51. Find six geometrical means between m and n.

52. Simplify $7\sqrt[3]{54}+3\sqrt[3]{16}-5\sqrt[3]{128}+\sqrt[3]{2}$ and $2\sqrt{\tfrac{5}{3}}+\sqrt{60}-\sqrt{15}+\sqrt{\tfrac{3}{5}}$.

53. Add together $\sqrt[3]{\tfrac{3}{2048}}$, $\sqrt[3]{1\tfrac{7}{9}}$, and $\sqrt[3]{93\tfrac{3}{4}}$; and $\sqrt{\frac{1}{3a^2b}}-\frac{3}{a}\sqrt[4]{\frac{a^4}{729b^6}}+\frac{1}{ab}\sqrt{3b}$.

54. Multiply $\sqrt{2pq}$, $\sqrt[3]{4q^2r}$, $\sqrt[4]{3p^3r}$ together ; and $\sqrt[8]{\frac{a^m}{b}}$, $\sqrt[12]{\frac{a}{bc}}$, and express the products with fractional indices.

55. What is the product of $(1+\sqrt{3})(2-\sqrt{3})(3+\sqrt{3})$; $(2\sqrt[3]{2}-3\sqrt{3})\times(2\sqrt{2}-3\sqrt[3]{3})$?

56. Expand $(2+\sqrt{3})^5$ and $(\tfrac{1}{2}\sqrt{7}-\tfrac{1}{3}\sqrt{3})^3$.

57. Simplify and square $\sqrt{a+\sqrt{b}}\pm\sqrt{a-\sqrt{b}}$.

58. Reduce to their simplest form $\sqrt{\sqrt{32}+\sqrt{24}}$ and $\sqrt{\sqrt{63}-\sqrt{35}}$.

59. Reduce to its lowest terms $\frac{9a^5+2a^3+4a^2-a+1}{15a^4-2a^3+10a^2-a+2}$.

60. What is the common difference in an arithmetical progression of 22 terms, whose first term is 1 and last 15 ?

61. Between 1 and 3 there are 10 terms of a geometrical progression : what is the second term of the series ?

62. It is required to find two numbers such that the product of the first by 17 shall exceed by 7 the product of the second by 26.

63. Expand by Binomial Theorem $(1+\tfrac{1}{5})^{-12}$, and prove that the third term in the expansion is the greatest.

64. Find the sum of $\frac{3h}{(h-2x)^2}+\frac{2h+x}{(h+x)(h-2x)}-\frac{5}{h+x}$.

65. Reduce to their lowest terms $\frac{30^{3n-1}b^rc^{r+2}-6a^{2n-4}b^3c^rd^{r-1}}{20a^nb^{r-1}c^2d^2-4a^{-3}b^2d^{r+1}}$ and $\frac{(a+b)(a+b+c)(a+b-c)}{4b^2c^2-(a^2-b^2-c^2)^2}$.

66. A person bought a number of sheep for 80*l*., and if he had bought four more for the same money he would have paid 1*l*. less for each. How many did he buy?

67. Sum the series $\frac{1}{8}+\frac{4}{8^2}+\frac{6}{8^3}+\frac{}{8^4}+\frac{1}{8^5}+\frac{4}{8^6}+\frac{6}{8^7}+\frac{3}{8^8}+\frac{1}{8^9}+\ldots$ in which the numerators 1, 4, 6, 3 repeat, but the denominators proceed according to the increasing power of 8.

68. What number of words of three letters can be formed of 8 consonants and 1 vowel, the vowel being the central letter?

69. Solve equations (i) $\frac{a}{a^2-b^2}+ax-2a+bx-b\,.\,\frac{a^2+ab-b^2}{a^2-b^2}=0$;

(ii) $\sqrt[m]{(a+x)}=\sqrt[2m]{(x^2+5ax+b^2)}$.

70. Employ the Binomial Theorem in the extraction of $\sqrt[5]{35}$.

71. Find the greatest common measure of a^2-b^2+6b-9 and $a^2-2ab+b^2+6a-6b+9$; of $2x^4-3x^3-28x^2+75x-91$ and $10x^3-41x^2+43x-77$.

72. Find the L. C. M. of x^2-y^2, $x^2-2xy+y^2$, and x^3-y^3; also of $p^2+8pq+15q^2$, p^2+3pq, and $3pq+15q^2$.

73. It is required to find a number such that if we multiply it by 5, the product will exceed 20 by as much as the number itself is less than 20.

74. What will a capital of 1200*l*. amount to at compound interest for 36 years at 4 per cent per annum?

75. If $a:b=4:5$, $d:f=5:2$, $e:c=6:7$, $d:b=7:3$, and $f:c=4:3$, by what would the proportions $a:b:c:d:e:f$ be numerically represented?

76. The diagonal of a rectangle whose breadth is 119 feet less than the length measures 221 feet: required the length and breadth of the rectangle.

77. What equation has the two roots $a+b-c\sqrt{-1}$ and $a+b+c\sqrt{-1}$?

78. By what number must the terms of the proportion 339 : 355 be increased or diminished so that the proportion may remain as 21 : 22?

79. Remove the irrational quantity from the denominator of $\frac{69+\sqrt{-3}-6\sqrt{-5}-7\sqrt{15}}{3-\sqrt{-3}+3\sqrt{-5}}$.

80. One root of the equation $x^2+444\frac{61}{60}x=975406\frac{5}{66}$ is $1234\frac{5}{6}$. Required the other.

81. When are the roots of the equation $x^2-ax+b=0$ both *positive*, when both *negative*; when is the greater *positive* and the less *negative*, and when the less *positive* and the greater *negative*?

82. There is a number consisting of two digits whose difference is 5, if the digits be inverted they will form a number equal to $\frac{3}{8}$ of the former: required the number.

83. Required the time between two and three o'clock when the hands of a watch are exactly opposite.

84. Simplify $\sqrt[3]{a^2}\,.\,\sqrt{a^{-1}}$.

85. Divide 27 into two numbers, so that the greater divided by the less may be to the less divided by the greater as 3 to $\frac{3}{4}$.

86. There is a field in the form of a parallelogram containing 235200 square yards: find its length and breadth, the former exceeding the latter by 140 yards.

87. $(x^2+xa)^{\frac{1}{2}}+(x^2-xa)^{\frac{1}{2}}=x$. Find the value of x.

88. $\frac{\sqrt{x}+\sqrt{a}}{\sqrt{x}-\sqrt{a}}=\frac{b}{a}$. Find the value of x.

89. What fraction is that which, if its numerator be increased by 7, will be equal to 2, but if its denominator be increased by 4, will equal $\frac{1}{2}$?

90. $\left.\begin{array}{l} x^2+y^2=m, \\ x^2-y^2=n. \end{array}\right\}$ Find the value of x and y.

91. A can do a piece of work in 10 days, B can do the same work in 15 days: how soon can they do it together?

92. An officer ranging his men in the form of a square has 59 men to spare; he then increases the side by 1 man, and finds himself in want of 60 men to complete the square: how many men had he?

93. Divide $x^4-\frac{1}{x^4}$ by $x-\frac{1}{x}$.

94. Divide $x-y$ by $\sqrt{x}-\sqrt{y}$.

95. A cask of wine by leakage has lost one-fourth of its contents; 20 gallons more have been drawn, and it is now found to be $\frac{5}{8}$ full: required the contents of the cask.

96. Two travellers start at the same time from separate places, 150 miles distant: in what time will they meet, walking respectively 7 and 8 miles a day?

97. A boy being asked how many marbles he had, said, if he had as many more, half as many more, and 5, he should have 130: how many had he?

98. A post is a third of its length in the mud, a fourth in water, and 5 feet above water: what is its length?

99. $\left.\begin{array}{l} x:y::a:b, \\ x^2+y^2=m. \end{array}\right\}$ Required the value of x and y.

100. There are two numbers in the proportion of $\frac{1}{5}$ to $\frac{1}{4}$, which, being increased respectively by 5 and 15, are in the proportion of $\frac{1}{18}$ to $\frac{1}{9}$: required the numbers.

101. Find the difference between $3x+\frac{x}{b}$ and $x-\frac{x-a}{c}$.

102. Find the sum of $\sqrt[3]{40}$ and $\sqrt[3]{135}$, and the difference between $\sqrt[3]{\frac{2}{3}}$ and $\sqrt[3]{\frac{9}{32}}$.

103. Find the sum of the numbers 1, 2, 3, 4, 5, &c. continued to 500.

104. $\sqrt{x}+\sqrt{a+x}=\frac{2a}{\sqrt{a+x}}$. Find the value of x.

105. A man engaged to work 50 days on these conditions, that for every day's work he was to receive 2*s.* 3*d.*, for every idle day he was to pay 9*d.*; at the end of the time he received 4*l.* 2*s.* 6*d.* How many days did he work, and how many did he play?

106. $\left.\begin{array}{l} x+y=a, \\ x^2-y^2=b. \end{array}\right\}$ Find x and y.

107. Expand $(x-2)^4$, $(a-b+c)^3$, and $(1+x)^6$.

108. A person lost by gambling $\frac{1}{6}$ of his money, and afterwards won 4 shillings; he then lost half the remainder, and won 4*s.* 6*d.*; he then had $1\frac{1}{2}$ guineas left: what had he to begin with?

109. Find the G. C. M. of x^2-1 and $xy+y$, of x^4-b^4 and $x^5+b^2x^3$, of x^3-b^2x and $x^2+2bx+b^2$, of x^2-y^2 and x^4-y^4.

110. Divide 210 acres of land among *A*, *B*, *C*: let the shares of the first two be to each other as 4 to 5, and let *C* have a third as much again as the other two together.

111. Find the G. C. M. of $x^3+2x^2y+x+2y$ and $2x^4-5x^3y+2x^2-5xy$, of $x+(x+b+c)y+b+c$ and $x+(x-b-2c)y-b-2c$.

112. Divide 191 into two such parts that if one be divided by 23 and the other by 15, the difference of the quotients may be 5.

113. Find the least common multiple of $3(a+b)$ and $6(a^2-b^2)$.

114. Simplify $\dfrac{\dfrac{x}{x+y}}{x+\dfrac{x-y}{x+y}}$.

115. Find the least common multiple of a^2b+ab^2, a^3-ab^2, and b^3+ab^2.

116. $\sqrt{m^2+x^2}=(n^4+x^4)^{\frac{1}{4}}$. To find x.

117. $\sqrt{x}+\sqrt{p+x}=\dfrac{2p}{\sqrt{p+x}}$. To find x.

118. The first digit of a certain number exceeds the second by 5, and if the number be divided by the sum of the digits the quotient will be 8; required the number.

119. *A* and *B* have the same income, *A* saves annually a fourth part of his, but *B* spends 40*l.* more than *A*, and at the end of 5 years finds he owes 75*l.*: required the income of each.

120. *A* can do a piece of work in 10 days; with *B*'s assistance for 4 days, he can do it in 7 days: how long will *B* be in doing the work himself?

121. Two persons, *A* and *B*, lay out equal sums of money in trade; *A* gains 140*l.*, and *B* loses 180*l.*, and *A*'s money is now double of *B*'s. What did each lay out?

122. How much foreign brandy at 8*s.* a gallon, and British spirits at 3*s.* a gallon, must be mixed together, so that by selling the mixture at 9*s.* a gallon there may be a gain of 30 per cent?

123. *A* and *B* together can perform a piece of work in 8 days, *A* and *C* in 9 days, and *B* and *C* in 10 days: in how many days can each perform the work alone?

124. Divide the number 196 into 4 parts, such that if the first be increased by 6, the second diminished by 6, the third multiplied by 6, and the fourth divided by 6, the sum, difference, product, and quotient shall be equal to one another.

125. $x-\sqrt{x}=a.$

126. $3x^{2n}-2x^{n}=25.$

127. I sold some goods for 24*l.*, and gained as much per cent as the goods cost me: what was the price of the goods?

128. I bought a number of cattle for 80*l.*, and if I had bought four more for the same money I should have paid 1*l.* less for each: how many did I buy?

129. Find four numbers in arithmetical progression such that the product of the extremes shall be 85, and that of the means 117.

130. Find three numbers in geometrical progression whose sum shall be 21, and the sum of their squares 189.

131. A farmer bought as many sheep as cost him 40*l.*, and sold them all except 10 for 44*l.*, gaining thereby 6*s.* a head: how many sheep did he buy?

132. A draper bought cloth for 21*l.*, which he sold at 1*l.* 17*s.* 11*d.* a piece, by which he gained as much as a single piece cost him: find the number of pieces.

133. The sum of four numbers in arithmetical progression is 30, and the sum of their squares is 270: what are the numbers?

134. Find four numbers in arithmetical progression whose common difference is 5, and their continued product 1056.

135. Simplify $\dfrac{(1-x)+\dfrac{1}{\sqrt{(1+x)}}}{1+\dfrac{1}{\sqrt{(1-x^2)}}}$, $\dfrac{1+\dfrac{\sqrt{(a^2-x^2)}}{\sqrt{(a^2+x^2)}}}{\sqrt{(a^2+x^2)}+\sqrt{(a^2-x^2)}}$.

136. One root of the equation $x^2-(5p-7q+9r)x+4p^2+18pr+18r^2=13pq+10q^2+27qr$ is $4p-5q+6r$: what is the other root?

137. Take the cube root of $7\frac{1}{5}$ and $9\frac{1}{6}$.

138. The extremes of an arithmetical series are 32·9 and 508·9, and the number of terms 41: what is the common difference?

139. Place 3 arithmetical means between 6 and 12, and 3 geometrical means between 27 and 2187.

140. Find x in the equation $\frac{1}{n}-\frac{1}{x}=\sqrt{\frac{1}{n^2}-\frac{1}{x}\sqrt{\frac{4}{n^2}-\frac{7}{x^2}}}$.

141. Sum the series $1+3x+6x^2+10x^3+15x^4+$ &c. and $\frac{1}{1.3}+\frac{1}{3.5}+\frac{1}{5.7}+\frac{1}{7.9}+$ &c.

142. Rationalise the denominator of the fraction $\frac{\sqrt{ab}}{\sqrt[4]{5}+\sqrt[4]{3}}$.

143. A father is now 49 years of age, which is 7 years less than the united ages of his three sons: when was his age exactly equal to theirs?

144. Simplify $(a-2b)x^3+(a+2b)x^3-(b-a)x^3-x^3$.

145. Divide $x-\frac{1}{2}(x-\frac{2}{3}y)$ by $y-\frac{1}{3}(x+\frac{3}{2}y)$.

146. Find the sum of $x^{\frac{5}{2}}-bx+\frac{b^2}{\sqrt{x}}-$ &c., *ad infinitum.*

147. What is the n^{th} term of the series $\frac{1}{2}+\frac{1}{6}-\frac{1}{6}-$, &c. and the sum of n terms?

148. Find two numbers whose difference is a times, and whose product is b times as great as their sum.

149. The reciprocal difference of two numbers, plus the reciprocal sum, is 3; the reciprocal difference of the numbers, minus the reciprocal sum, is 1: what are the two numbers?

150. What will be the solution of the preceding question, if for 3 and 1 we substitute the general expressions a and b?

151. Given $\left.\begin{aligned}x^3y+y^3x&=512500,\\ x^2y-y^2x&=5500.\end{aligned}\right\}$ Find x and y.

152. Find three numbers, such, that if to the square of each the product of the other two be added, the sums shall be squares.

153. What is the exponent for a geometrical progression consisting of 32 terms, the first of which is 5, and the last 80? Find also the sum of the progression, and its 20th term.

154. $\frac{(x^{6\frac{1}{2}}-x^{5\frac{1}{2}})(x^2-x)}{8}+\frac{x^{4\frac{1}{2}}-x^{3\frac{1}{2}}}{x-2}-\frac{5x^{6\frac{1}{2}}(x^2+1)-8x^{3\frac{1}{2}}}{40}=\frac{5x^{8\frac{1}{2}}-x^{7\frac{1}{2}}}{4}$: find x.

155. Take the square root of ·00789, and the cube root of 15⅝.

156. Simplify $\frac{a+\sqrt{-b}}{a-\sqrt{-b}}$ and $\frac{b}{\sqrt[n]{\{a-\sqrt{(a^2-b^n)}\}}}$.

157. Two cog wheels, with 15 and 28 teeth respectively, work into one another: if the first make 16 revolutions in 7½ seconds, how often will the second revolve in 21 seconds?

158. Simplify $\sqrt{\left(\frac{a^{2x}}{a^3}\right)^x}\cdot\sqrt[x]{\left(\frac{a^{5x}}{a^9}\right)^x}\cdot\sqrt[x]{\left(\frac{a^{12}}{a^{6x}}\right)^x}$.

159. What is the present value of an annuity of 22*l.*, to be paid for 21 years, 4 per cent compound interest being allowed?

160. $\left.\begin{aligned}&\frac{3x+2y}{5}-\frac{5x-\frac{3y}{4}+1}{3}=x+\frac{y-2x}{10}-\frac{4x-y}{7},\\ &y+2x:y-2x=12x+6y-3:6y-12x-1.\end{aligned}\right\}$

161. Divide $\sqrt[3a-2]{(x^{5am})^3\,.\,(x^6)^2}$ by $\sqrt[3a-2]{x^{10m}\,.\,x^{18a}}$, and $\sqrt[mx]{\frac{a^{x-2}b^{x-4}}{c^{x-6}}}$ by $\sqrt[mx]{\frac{b^{-4}}{a^2c^{-6}}}$.

162. A man has two pieces of iron whose weight he desires to know; he knows that ⅖ of the first piece are 96 pounds lighter than ¾ of the other piece, and that ⅝ of this other piece weigh as much as 4/9 of the first: required the weight of each piece.

163. The first two terms of an arithmetical progression added together =6, the next three =24: how many terms will make 42?

164. In how many years does a capital double itself at 5 per cent?

165. In how many years will a capital of 2709*l.* be as great as that of 3815*l.* in 7 years, the interest of both being 3½ per cent?

166. At how much per cent must a capital stand out in order to double itself in 10 years?

167. Form an equation which shall have the two roots $a+b-c\sqrt{-1}$ and $a+b+c\sqrt{-1}$.

168. One root of the equation $x^2+444\frac{5}{6}\frac{1}{6}x=975406\frac{5}{36}$ is $1234\frac{5}{6}$: required the other root.

$$169.\quad \left.\begin{aligned} \frac{x}{3}+\frac{y}{5}+\frac{2z}{7}&=\ 58,\\ \frac{5x}{4}+\frac{y}{6}+\frac{z}{3}&=\ 76,\\ \frac{x}{2}+\frac{3z}{8}+\frac{u}{5}&=\ 79,\\ y+z+u&=248.\end{aligned}\right\}\quad \text{Find } x,\ y,\ z,\ u.$$

170. Find the approximate value of $\sqrt{2}$.

171. Divide the number 30 into three such positive integral parts x, y, z, that $2x+9y+15z$ may $=419$.

172. How many combinations can be made of 60 things, taken 2, 3, 4, or 5 together?

173. Divide $\frac{a^7b^2}{5}-\frac{47a^6b^3}{40}+\frac{9a^5b^4}{2}-12a^4b^5$ by $\frac{2a^3b^2}{5}-\frac{3a^2b^3}{4}+6ab^4$, and $\frac{a^3c}{b^5}+\frac{a^4c}{b^4}-\frac{7a^5c}{b^3}-\frac{3a^6c}{b^2}+\frac{a^2c^3}{b^2}-\frac{2a^3c^3}{b}-a^4c^3$ by $\frac{a}{b^3}+\frac{3a^2}{b^2}+c^2$.

174. Find the product of $\sqrt[6]{a^5}$ and $\sqrt[8]{a^7}$, and resolve $\sqrt[x]{a^{nx+m}}$ and $\sqrt[3]{a^{14}}$, into quantities partly rational and partly surd.

175. Resolve $\sqrt[n]{\frac{a^{n+1}}{b^{n-2}c^{n-3}d^{n-4}}}$, and take the root of $\sqrt[4x+6y]{\frac{a^{28xz}\cdot a^{10yz}}{a^{48yz}}}$.

176. Reduce $\frac{xy+2x^2-3y^2-4yz-xz-z^2}{9xz+2x^2-5xy+4z^2+8yz-12y^2}$ to its lowest terms.

177. $\{12(13580-x)-9\}^2+\{5(13580-x)-1\}^2=\{13(13580-x)-8\}^2$: find x.

178. Find the least common multiple of $a+b$, a^2-b^2, and $(a-b)^2$.

179. In how many ways can we write a^2b^2c, and how many combinations, 3 and 4 together, can be made with 90 things?

180. Required two numbers such that the sum and product of them together are 34, and the sum of their squares exceeds the sum of the numbers themselves by 42.

181. What formulæ will express the numbers sought in the preceding question, if we substitute a for 34, and b for 42?

182. (i) $x=\sqrt{a^2+x\sqrt{b^2+x^2}-a^2}+a$, (ii) $\frac{1}{n}-\frac{1}{x}=\sqrt{\frac{1}{n^2}-\frac{1}{x}\sqrt{\frac{4}{n^2}-\frac{7}{x^2}}}$.

183. (i) $5\sqrt[3]{\frac{4}{25}x^2-\frac{8}{125}x^3}+2x=1\frac{2}{3}$, (ii) $\sqrt[2m]{m^2x^2-mnx}=\sqrt[m]{mx-n}$.

184. (i) $\frac{n-p\sqrt[m]{x-r}}{s-t\sqrt{x-r}}=u$, (ii) $\frac{50\sqrt[10]{x+24}-9}{3+5\sqrt[10]{x+24}}=7$.

185. What is the value of $\frac{\sqrt{2a^3x-x^4}-a\sqrt[3]{a^2x}}{a-\sqrt[4]{ax^3}}$, when $x=a$?

186. A person drew a certain quantity of wine from a vessel containing 81 gallons, and then filling the vessel with water, drew the same quantity of the liquid as before: this was repeated altogether four times, and it was then found that there were 16 gallons of pure wine left. How much was taken at each draught?

187. There are two pipes, one of which will fill a cistern in the time p, and the other in the time q: in what time will they both fill it?

188. Suppose that p pounds of gold out of water weigh q pounds in water, and that r pounds of silver weigh s pounds in water; suppose also that a mass of gold and silver weighing a pounds, when weighed in water weigh only b pounds: find the quantities of gold and silver in the mass.

189. Express 112521 in the scale of 12, and find the square root in that scale.

190. Two men start in opposite directions from places 154 miles apart; one travels at the rate of 3 miles in 2 hours, and the second 5 miles in 4 hours: how long and how far will each travel before they meet?

191. A man bought a horse for a certain sum, and sold it again for 144*l.*, and gained just as much per cent as the horse had cost him: find the cost of the horse.

192. A certain company at an inn had 7*l.* 4*s.* to pay; whereupon two of the company having left, the rest were obliged to pay 1*s.* a-piece more than they should have done: what was the number of persons?

193. If $p, q, r, s,$ &c. be a series in geometrical progression, then will $\frac{1}{p^2-q^2}, \frac{1}{q^2-r^2}, \frac{1}{r^2-s^2},$ &c. be in geometrical progression.

194. If $a : b :: b : c$, and $b : c :: c : d$, show that $a : d :; a^3 : b^3$.

195. There are three numbers in geometrical progression whose sum is $6\frac{1}{2}$, and the sum of the first and second is to the sum of the second and third as 1 to 3: find the numbers.

196. A grazier bought a certain number of oxen for 80 guineas: if he had bought four more for the same money, they would have cost him a guinea a-piece less: what was the number of oxen?

197. There is a certain number consisting of three digits in arithmetical progression; if this number be divided by the sum of its digits, the quotient will be 48; and if from the number be subtracted 198, the digits will be inverted: what is the number?

198. A schoolboy has a geometrical proportion between four numbers; he desires to reduce the numbers of the proportion, and for this purpose subtracts from each of the four terms the same number, and obtains by so doing the false proportion $41 : 93 = 7 : 51$. What was the original proportion?

199. There is a certain floor in the form of a rectangle whose dimensions are such, that if it had been 2 feet broader, and 3 feet longer, it would have been 64 square feet larger; but if, on the other hand, it had been 3 feet broader and 2 feet longer, it would then have been 68 square feet larger: required the dimensions of the floor.

200. By what number must each of the factors of the two unequal products $52 \cdot 45$ and $66 \cdot 37$ be diminished, so that the new products may be equal?

201. We find that 37 lbs. of tin weighed in water lose 5 lbs. weight, and 23 lbs. of lead also weighed in water lose 2 lbs.; supposing we have a mixture of lead and tin which weighs 120 lbs. in the air, and 106 in water; of how many lbs. of tin and lead does the mixture consist?

202. A merchant buys two pieces of cotton, together 36 yards long; each piece costs per yard as many pence as there are yards in the piece, and the prices of the two pieces are in the proportion of 4 : 1. Required the length of each piece.

203. Find two numbers whose sum is 187, the first of which divided by 9 leaves 1 remainder, and the second when divided by 13 leaves also 1 remainder.

204. If the population of a place increases $\frac{1}{30}$ annually, how many will 10000 number after the lapse of a century?

205. Two travellers, R and S, start at the same time from two places to meet each other; when they met it appeared that R had accomplished d miles more than S; if, now, R, continuing the journey at the same pace, finishes it in a days, and S pursuing his path at his former pace can finish his journey in b days, how many miles are these places asunder, allowing $d=30$, $a=4$, and $b=9$?

206. Solve the equation

$$nx=py=\tfrac{1}{2}\sqrt{(m+x+y)(m+x-y)(m-x+y)(-m+x+y)}.$$

207. In how many different ways can 52 cards be distributed among four whist players, so that each may receive 13?

208. Find the value of $x+2(a+b)\sqrt{3(a^2+b^2)+x}+10ab$ when $x=(b-3a)^2-3(a^2+b^2)$.

209. Simplify $\frac{(5\sqrt{7}+6\sqrt{10})}{(5\sqrt{1.75}+6\sqrt{2.5})}$ and $\frac{13\sqrt{15}-7\sqrt{21}}{13\sqrt{1\frac{2}{3}}-7\sqrt{2\frac{1}{3}}}$.

210. What annuity, to continue 20 years, may be purchased for 1000*l.* at $3\frac{1}{2}$ per cent? Write down the formula for this.

ANSWERS TO EXAMPLES.

EXERCISE I.

[1] 9. [2] 30. [3] −16. [4] −3. [5] 10. [6] 7.
[7] 1. [8] 6. [9] 31. [10] 23. [11] 32. [12] 76.

EXERCISE II.

[1] 8. [2] 2. [3] 9. [4] 1. [5] 5. [6] 6.
[7] 6. [8] 11. [9] 16. [10] 43. [11] 168. [12] 14.

EXERCISE III.

[1] 11. [2] 8. [3] 13. [4] 35. [5] 16. [6] 14.
[7] 16. [8] 15. [9] 53. [10] 9. [11] 36. [12] 28.

EXERCISE IV.

[1] 21, 17, 21, 17, and 4904, 4330, 4904, 4330. [2] 2, 4, 6, 8, 8.
[3] 179, 427, 427, 631, 541, 155, 427. [4] $m-(n+p)-q=22$.
[5] $m+n-p+q-r+x+y$. $m-n+(p-q)=44$.
$m-\{(n+p)-q\}=66$.
[6] 4000, 4199, 1600, 179, 1247, 179, 47, 1284, 179, 284.
[7] $m+n+(m+n)p-(m+n)q-p+q$.
[8] 120, 120, 840, 840, 840, 3200000.
[9] 15435, 79380, −9975, −9975, 5, 25.

EXERCISE V.

[1] $a+b$. [2] $a-b$. [3] $a+b+c$. [4] ab.
[5] abc. [6] $\frac{a-b}{a+b}$. [7] $\frac{a+b+c}{a}$. [8] $\frac{xyz}{x+y+z}$.
[9] $\frac{abcd}{a+b}$. [10] No; thus, $\frac{a+b+c+d}{c+d}$.

[11] Because the quantities are not written in the order of succession; thus, $abxy$.

[12] $(a+b)c$. [13] $(a-b)c$. [14] $ab+cd$. [15] $ab-cd$.
[16] No; thus, $\frac{abcd}{a+b+c+d}$. [17] $\frac{xy+pq}{x+y}$. [18] $\frac{6(a+b)}{c+d}$.
[19] $\frac{8(a-b)}{cd}$ or $\frac{8(b-a)}{cd}$. [20] a^2b^2. [21] $\frac{abc}{a+b}-c$.

[22] 20. [23] 5. [24] 114. [25] $\frac{5}{8}, \frac{5}{16}, \frac{5}{8}, \frac{5}{90}$.
[26] 10. [27] 90. [28] 17*l.* 10*s.* [29] 144*l.*
[30] Exactly. [31] 39*l.* [32] 2*s.* 6*d.* [33] $30\frac{4}{5}$.
[34] 12880*l.* [35] 12362*l.* [36] $1\frac{1}{2}$. [37] $13\frac{3}{4}$ years.
[38] $49\frac{1}{4}$ years. [39] 1565*l.* [40] 488*l.*

EXERCISE VI.

[1] $3a$. [2] $24a$. [3] $10b^2$. [4] $3a^2b$. [5] $3d$.
[6] $11x$. [7] $5a-b+2c-4d+6e$. [8] $2(m-2)$. [9] $3(a-x)$.
[10] $a-y$. [11] $(a+b)x$. [12] $8a-8b+4c-17d+4$.
[13] $9h+5c-9f-6g+5x$. [14] $23a+8b+7c-11d-8e-2h$.
[15] $10a^2+13bc-4c^2+6f+3$. [16] $4g+4h-2k+10l-6$.
[17] $13x-11y+3z+20v$. [18] $4p+3q+2v+15s$.
[19] $15a+3b+2c+5d+e$. [20] $a^3+3a^2b-5ab^2-5b^3-x^2-xy+9y^2$.
[21] $(3a+15c)x+(4b-3d-1)y-4z$. [22] $-6\sqrt{x}+3\sqrt{y}+20\sqrt{z}$.
[23] $16\frac{1}{4}a^3x-8a^2x^2+8a^2x^3$.
[24] $x^3+9x^2y-2xy^2-8y^3-19x^2z+8xz^2-(x-1)z^3$.

EXERCISE VII.

[1] $-8a, -6ac, -8ab, -11xyz$.
[2] $-11(a+d), -7(x+y+z), -17a(bc+dc), -2a(b+c)$.
[3] $2a, 4a, 6b, 8x$. [4] $10ac, 12abc, 14(x-y), 16(x+y+z)$.
[5] $18a, 47ab, 12ab, 15abcd, 17(a+e), 17(x+z)$.
[6] $8a+b, 9x+15y$. [7] $xy+4z, 12ab+ac$.
[8] $-4a, -6b, -16ac, -32x, -3abc, -4(x+y)$.
[9] $a-b, 5b-5c, 2x-4y, xy-yz, xy-5uv$.
[10] $4, a, b, 3a, 6ab, -7xyz, -3(x+y)$.
[11] $2a+3b, 2ab+3c, 5xy+5z, 2xyz+4a, 3abc+4d, 6a^2+4b^2$.
[12] $2b$ or $-2b$. [13] $8b$ or $-8b$.
[14] $a+2b-2c, -b+7c, -2x+2z$.
[15] $5b-3a, 6y-5x, 7x-3z, 3cd-5ab, -5xy-3ac, 3ab+5xy, 12xyz+5x$. [16] $7a, 8b, 21ac, 14xyz$.
[17] $(a-b)(x+y), (4-x)(a+b), (3a-4b)(b+c), (5p+3q)(q+r), (6a+5d)(c+d)$. [18] $6a+2b+2f, -6a-2b-2f$.
[19] $2a-4b+4c, -2a+4b-4c$.
[20] $11a+5b-3c-7d-5e+3f+11g, 21a-5b+3c+5e-3f-11g$.
[21] $6x+9y-12z, -6x-9y+12z$. [22] $16a-7b$. [23] $-c+24d$.
[24] $-17c-12ef$. [25] 0. [26] $20x+3y$.
[27] $2b$. [28] $(a-b+c)(x+y)$.
[29] $-6a^3-6a^2b+2ab^2-b^3=-(6a^3+6a^2b-2ab^2+b^3)$.
[30] $(5x-10y)\sqrt{y}-(11-2x)\sqrt{xy}-12a$. [32] $-18a^2+13a-14$.
[31] $(13a-17b)\sqrt{(x-y)}-17b\sqrt{(x+y)}-(a+b)xy+x^3-a^2y$.

Exercise VIII.

[1] $2a^2+2b^2$. [2] 0. [3] $x^2y+2ab^3-3d^2$. [4] $-c-5d^2x$.
[5] $4ayz+p^2r$. [6] $7mn^2-5y$. [7] $21def-9a^2x^2$. [8] $4b^4$.
[9] $24b^2e^4-cx^3-10m^2p$. [10] $8a^2bc^4-20b^3$. [11] 0.
[12] $b^3z^5-2p^2q^2+1$.

Exercise IX.

[1] $(a+b)x+(3a^2+4b^2)x^2$. [2] $(a+b+c)x^3$.
[3] $(a+b)x+(p+q+r)x^2$. [4] $(m+n-p)xy+(b+d)y^2$.
[5] $(a+b-c)x+(a+b-c)y$. [6] $(xy+x^2+y^3)z+(a-x)z^3+(x^2-x)y^2$.
[7] $(b^2-a^3c+c^2)x+(a-b^2)cy$. [8] $(p+a)x^2y^2-(q^2+a^3-p)xy^2$.
[9] $(m-a)nx-(b-c^2)y-(n-x)yz^2$.
[10] $(a^2bc^3-d)x+(b^3-q)c^2+(a-p^2)x^3$.
[11] $(4a+1)b^2-(3c+5a^2)yz+(2b-d)x$.
[12] $(ac+3b)(x+y)+2a(b-2x)$. [13] $\{(5-a)be^3+4a\}(x+1)$.
[14] $2y(3x^2-bc)(p-q)+(a-2x)y^2$. [15] $4a^2b^2(c^2-5)+12b^3(c^2+a)$.
[16] $\{(mn+x)-3a(b+y)\}(y^2-z^2)$. [17] $(2a^2+3ac^3)b(y^3-1)$.
[18] $7(x-y)^3+(3a-bc^2)(x+y)+ab(x^3-1)$.

Exercise X.

[1] $27ab+36b^2$. [2] $-30ab-6ad+18ae$. [3] $3a^3b+3abc^2-3abd^2$.
[4] $a^2+2ab+b^2$. [5] a^2-b^2. [6] $a^2-2ab+b^2$.
[7] $-5a^4b^2-10a^3b^3-5a^2b^4=-(5a^4b^2+10a^3b^3+5a^2b^4)$.
[8] $a^4-2a^2b^2+b^4$. [9] $x^{10}-y^{10}$.
[10] $-6x^2+17xy-12y^2=-(6x^2-17xy+12y^2)$.
[11] $2x^3-7x^2y+5xy^2+2y^3$.
[12] $225a^4+330a^3b-299a^2b^3-308ab^3+196b^4$.
[13] $3x^7-11x^6+7x^5+11x^4-2x^3+x^2-28x+15$.
[14] $3a^5+4a^3b^2+4a^2b^3+ab^4+4b^5$.
[15] $6a^6b^3-27a^5b^4+44a^4b^5-38a^3b^6+3a^2b^7+2ab^8$.
[16] $\frac{1}{2}x^5-\frac{5}{72}x^4y-\frac{3}{16}x^3y^2+\frac{17}{60}x^2y^3-\frac{9}{40}xy^4+\frac{1}{20}y^5$.
[17] $x^7-2\frac{1}{6}x^6y+7x^5y^2-10\frac{1}{3}x^4y^3+12\frac{2}{3}x^3y^4-8\frac{2}{3}x^2y^5+6xy^6-2y^7$.
[18] $0{\cdot}2m^6-0{\cdot}46m^5n+0{\cdot}32m^4n^2-0{\cdot}8m^3n^3-2m^2n^4+0{\cdot}2mn^5+1{\cdot}12n^6$.
[19] $10x^3-15x^2-220x+525$.
[20] $a^6-8a^5-4a^4+100a^3-216a^2+612a-85$.
[21] $-x^4-y^4-z^4+2x^2y^2+2x^2z^2+2y^2z^2$.
[22] $14x^9+21x^8-6x^7+38x^6+7x^5+4x^4-30x^3+26x^2-36x-20$.
[23] $x^2+ax-bx-ab=x^2+(a-b)x-ab$.
[24] $x^2-ax+bx-ab=x^2-(a-b)x-ab$.
[25] $x^3-(a+b-c)x^2+(ab-ac-bc)+abc$.

[26] $x^4-(a-p)x^3-(b^2+ap+q^2)x^2-(b^2p-aq^2)x+t^2q^2$.
[27] $x^4-(3a-5b)x^3+(6a^2-15ab+8b^2)x^2+6ab(ba-4b)x+48a^2b^2$.
[28] $x^6-3x^4a^2+3x^2a^4-a^6$. [29] x^6-y^6. [30] x^7+y^7.
[31] $x^{2p}+(y^r-y^s)x^p-y^{r+s}$.
[32] $6x^{8n+2}-23x^{7n+1}+18x^{6n}-x^{5n-1}-3x^{4n-2}+4x^{3n-3}-x^{2n-4}$.
[33] $x^n+x^{n-5}y^5$.

EXERCISE XI.

[1] $6x^3+36x^2+178x+479$. [2] $-7x^3-15x^2+20x+179$.
[3] $8\frac{1}{3}x^4-362\frac{2}{3}x^3-4339\frac{1}{3}x^2-22825x+44947$.
[4] $-8x^5+5x^4-15x^3+46x^2-247x+227$.
[5] $\frac{1}{12}(588x^4-7804x^3+40078x^2-94688x+82440)=46\frac{1}{2}x^4-650\frac{1}{3}x^3+3339\frac{5}{6}x^2-7890\frac{2}{3}x+6870$.
[6] $36a^4+2(17a-9b)x^3+(14a^2-48ab-7b^2)x^2+2ab(9a+23b)x-(35a^2-b^2)b^2$.
[7] $7\frac{1}{3}x^4-22\frac{2}{3}x^3a+36x^2a^2-57\frac{1}{3}a^3x+396\frac{2}{3}a^4$.
[8] $\frac{1}{12}(20x^5-240x^4+313x^3+1413x^2+911x-2025)$. [9] 0.

EXERCISE XII.

[1] $ab(a-b)$. [2] $-ab(a+b+1)$. [3] $(3a-2b)(5c-d)$.
[4] $(2a^2-b)(2a^2-b)$. [5] $y(y-1)(y-1)(y-1)$. [6] $(x+14)(x+15)$.
[7] $(x-y)(x+y)(x-z)$. [8] $(x+4)(x+9)$. [9] $(x+1)(x+1)$.
[10] $(x+4)(x-3)$. [11] $(x-9)(x+5)$. [12] $(x+9)(x-20)$.
[13] $\frac{1}{4}x^2(x-\frac{1}{2})(x+\frac{1}{2})$. [14] $9a^2(a+2)(a-1)$. [15] $(x+y+z)(x+8)$.
[16] $(a+b-c+d)(a+b+c-d)$. [17] $(b-c)(m+n+xy)$.
[18] $b(2a-3b)(4a^2+6ab+9b^2)$. [19] $x(x-1)(x+1)$.
[20] $(a^2+ab+b^2)(b-c)$. [21] $(x+y)(x-y)(-x+y)$.
[22] $(a-b+c)(a-b-c)$. [23] $2a(a^2+1)(a+2)$.
[24] $(a+b+c)(a+b-c)(a-b+c)(a-b-c)$. [25] $(2a+5b)(3a-b)$.
[26] $2(3a-2b)(3a+2b)$.

EXERCISE XIII.

[1] $2b$. [2] $3ac$. [3] $-5ac$. [4] $11x^2y$. [5] $13x^{p-r}y^{q-s}$.
[6] $13a^{p-r}b^{-q}c^{1-s}$. [7] a^3b^3c. [8] $-7a^2c$. [9] $9x^2y^2z$.
[10] $-5a^{-1}b$. [11] $-1\frac{1}{2}ab^{-1}c^{-2}$, or $-\frac{3a}{2bc^2}$. [12] $-\frac{2}{5}acd$.
[13] $-96a^2b^2ce$. [14] $\frac{1}{5}a(x+y)^3z$. [15] $3a^{-3}$.
[16] $-4b^{2q}c^{1-r}$. [17] $-\frac{1}{9x^3y^5}=-\frac{1}{9}x^{-3}y^{-5}$. [18] $(a+b)^{x-y}$.
[19] $\frac{1}{5}(x-y)$. [20] ab^{-1}, $\frac{2}{5}ab^{-1}$, $3ab^{-2}$, $a^{-3}b^{q-r}$, $ab^{-1}(x+y)(x-y)^{-1}$.

EXERCISE XIV.

[1] $3a-b-\frac{b^2}{2a}$. [2] $-a^2+1\frac{2}{3}ab-1\frac{1}{3}b^2$. [3] $7x-3y+\frac{y^2}{5x}+\frac{y^3}{5x^2}$.

[4] $-\left(x^2-7xy+1\frac{1}{2}y^2+\frac{8y^3}{3x}\right)$. [5] $\frac{1}{2}+\frac{b^2}{2a^2}+\frac{c^2}{2a^2}=\frac{1}{2}\left(1+\frac{b^2}{a^2}+\frac{c^2}{a^2}\right)$.

[6] $-\frac{1}{2}a(x+y)^2+\frac{1}{2}b(x+y)-\frac{xy}{4a(x+y)^2}$. [7] $\frac{7}{12}(x+y)+\frac{2a}{8b}(x-y)-\frac{3b}{4a}$.

[8] $5c^2d$. [9] $8a-5$. [10] $3a^2-2ab-b^2$.

[11] $a^4-3a^3b-5a^2b^2+7ab^3-2b^4$. [12] $x^4+x^3y+x^2y^2+xy^3+y^4$.

[13] $x^4-x^3y+x^2y^2-xy^3+y^4$. [14] $3a-2b-x$. [15] $2x^2+5x-7$.

[16] $x-y$. [17] $x^3-x^2y+xy^2-y^3$. [18] $5a+2$.

[19] $4a^3-5a^2b-8ab^2+2b^3$. [20] $b^3+(3a^2-c^2+3ac)b+a^2c$.

[21] $a^2+(2b-2c)a-bc$. [22] $-a^4-(2b^2-c^2)a^2-(b^2+c^2)b^2$.

[23] $3x^{4n+1}-4x^{3n}+2x^{2n-1}-x^{n-2}$. [24] $m^2(5m^2-2mn+4n^2)$.

[25] $2c^2+3cd-d^2$. [26] $a^3-a^2b+(b^2+x^2)a-(b^3+bx^2)+\frac{b^2(b^2+x^2)+x^4}{a+b}$.

[27] $a+b-\frac{b^2(a-b)}{a^2+b^2}$. [28] $\frac{4}{7}a^3-\frac{8}{7}a+\frac{2}{7}a\times\frac{1}{a^2+2}=\frac{2}{7}a\left(2a^2-4+\frac{1}{a^2+2}\right)$.

[29] $2x^{-1}ya^{-3}(2x^{-1}+ya^{-1})=4x^{-2}ya^{-3}+2x^{-1}y^2a^{-4}$.

[30] $x^{-2}y^2(3x^{-2}-3x^{-1}y+2y^2)=3x^{-4}y^2-3x^{-3}y^3+2x^{-2}y^4$.

[31] $\frac{5}{6}a^3-\frac{2}{3}a^2x+\frac{1}{4}ax^2-\frac{4}{5}x^3$. [32] $1{\cdot}2x^2+1{\cdot}8xy-1{\cdot}5y^2$.

[33] a being substituted for $(x+y)$, the quotient is $4a^2-6az+8z^2$; restoring $(x+y)$, it is $4(x+y)^2-6(x+y)z+8z^2$.

[34] $(x+y)^2-\frac{3}{2}(x+y)z-6\frac{3}{4}z^2-\frac{5(x+y)^2z^3-62(x+y)z^4+127z^5}{8(x+y)^3-12(x+y)^2z+16(x+y)z^2-20z^3}$.

[35] $2(x+y)^{-3}z^{-1}-5(x+y)^{-2}z^2-3(x+y)^{-1}z^{-3}$.

[36] $1\frac{1}{4}a^3-\frac{3}{8}a^2b+1\frac{7}{8}ab^2+1\frac{23}{32}b^3-\frac{b^4(502a^2-331ab-124b^2)}{32(4a^3-2a^2b+3ab^2-4b^3)}$.

[37] $x^2-2xz+y^2$. [38] $4x^2-2ax+3ab$.

[39] $4a^3-5a^3b-3ac^2+6c^3$. [40] $\frac{b}{b-1}a^2+ab^2-ca^{-1}$.

EXERCISE XV.

[1] $x=5$. [2] $x=a+4b$. [3] $x=c\pm\frac{b}{a}$. [4] $x=\frac{3b-4d}{a-2c-1}$.

[5] $x=300$. [6] $x=\frac{3ab}{a-b}$. [7] $x=9$. [8] $x=12$.

[9] $x=139\frac{1}{2}$. [10] $x=6$. [11] $x=30$. [12] $x=\frac{b^2(b+a)}{a(b-a)}$.

[13] $x=\frac{2cd}{c+3d}$. [14] $x=9$. [15] $x=33$. [16] $x=\frac{1}{2}a\times\frac{3a-5b}{a+4b}$.

[17] $x=79$. [18] $x=13$. [19] $x=42$. [20] $x=56$.

[21] $x=r\times\frac{p^2-q^2}{p^2+q^2}$. [22] $x=\frac{1}{2}(a-1)$. [23] $x=66\frac{2}{3}$.

[24] $x=\frac{cd}{b}\times\frac{a^2-bc}{b^2-c(c+ad)}$. [25] $x=2{\cdot}010424$. [26] $x=-519{\cdot}67567$.

[27] $x=\frac{e(ad+bc)}{de-cf}$. [28] $x=\frac{bdf(g+h)}{adf+b(ed+cf)}$. [29] $x=\frac{ac(1-3ab)}{c-ad}$.
[30] $x=\frac{ab-1}{bc+d}$. [31] $x=\frac{4a-3b}{3a-2b}$. [32] $x=\frac{3a-6}{4}=\frac{3}{4}(a-2)$.
[33] $x=\frac{c(a+d)(b-c)}{b(b-c)-a^2}$. [34] $x=d+c$. [35] $x=a+b+c-d$.
[36] $x=\frac{adfh+bcfh+bdeh+bdfg}{bdfhk}$. [37] $x=a\times\frac{b+c}{a^2-(a-c)c^2}$.
[38] $x=\frac{b(7a-2b)}{3(b-2a)}$. [39] $x=1$. [40] $x=\frac{14a^2-39ab}{9b-27ab-12}$.

Exercise XVI.

[1] 15. [2] 60 yards. [3] 45 yards. [4] 120*l*. [5] 36.
[6] 71 years. [7] 24. [8] 2 apples.

[9] The hands will come together 11 times, and $5\frac{5}{11}$ min. after 1 o'clock, $10\frac{10}{11}$ min. after 2 o'clock, $16\frac{4}{11}$ min. after 3 o'clock, and so on, $5\frac{5}{11}$ min. later every hour. [10] 400 lbs. [11] 840 eggs. [12] 5 yards.

[13] 48 min. past 10. [14] 17 and 434. [15] 96 marbles.
[16] $105\frac{1}{3}$ and $131\frac{2}{3}$. [17] A 270*l*., B 360*l*., C 540*l*.
[18] $\frac{mpa}{mp+np+nq}$, $\frac{npa}{mp+np+nq}$, $\frac{nqa}{mp+np+nq}$.
[19] 22 men, 18 women, and 50 children. [20] 7, 15, 48.

[21] 24 apple-trees, 7 pear-trees, 9 cherry-trees, 8 currant, and 15 gooseberry-bushes. [22] 252 cubic feet O, 1008 cubic feet N.

[23] The first 90, the second 100, and the third 480 quarts.

[24] 256 geese. [25] 50 years. [26] 28. [27] 12 sheep.

Exercise XVII.

[1] $(abx)^2$, $(5pq)^3$. [2] $(10bxy)^4$. [3] $\left(\frac{a}{b}\right)^2$, $\left(\frac{a+b+c+d}{a}\right)^3$.
[4] $\left(\frac{a+b+cd}{x-y}\right)^4$. [5] $10a^2+b$. [6] $(a^2b^3)^4$.
[7] $(p^2+q^3)^4$, $\left(\frac{s^3}{r^2}\right)^4$. [8] a^2+b^2, $(a+b)^2$, $\frac{x^2-y^2}{(x-y)^2}$. [9] $\frac{x^3-y^3}{(x-y)^3}$.
[10] $(a+b)^2+\sqrt{a+b}$. [11] $(p+q)^2$, $\sqrt{p+q}$. [12] $\sqrt[3]{15ab^4-c}$.

[13] 256, 625, 1296, 5184, 9025, 9801, 10201, 15625, 30976, 35721, 39601. [14] $9a^2b^2$, $225x^4y^2$, $121b^2c^2d^2$, $9a^4b^4$.

[15] $36x^4y^2$, $6\frac{1}{4}a^2b^2c^2$, $25a^2x^4y^2z^4$, $\frac{1}{9}a^2b^2$, $\frac{1}{36}x^2y^4$, $10\frac{6}{25}a^6b^2$, $25a^2x^6y^4$.
[16] a^4, a^{10}, a^{16}. [17] a^n, x^{2p}, a^{3r}, a^{6r}, a^{pr}. [18] x^{mn}, x^{pn}, x^{ns}, x^{2m}, y^{3m}.
[19] $3a^2b^4c^2$, $16a^6b^2c^2$, $256a^6b^2c^2$, $11x^4y^6z^8$, $121x^4y^6z^6$.
[20] 324, 36, 2304, 1100, 6912, 57600, 921600.
[21] a^4b^2, $b^2x^4y^2$, $2250x^4$, $768x^4y^4$, $\frac{a^2}{c^2}$
[22] 225, 625, 625, 3025, 3025, 15625, 18225, 18225.

[23] $a^2+2ab+b^2$, $a^2-2ab+b^2$, a^2-2a+1, a^2+2a+1, $x^2+2xy+y^2$, $x^2-2xy+y^2$, x^2-2x+1, x^2-6x+9, x^2+4x+4, x^2+6x+9, $b^2+2bc+c^2$, $c^2+2cd+d^2$. [24] $x^2+4xy+4y^2$, $4x^2-4xy+y^2$, $9+6y+y^2$, $x^2-8x+16$, $x^2-12x+36$, $4x^2+24xy+36y^2$, $a^2-2ay+y^2$, $16x^2-40ax+25a^2$, $36a^2+84ab+49b^2$, $x^2y^2+2xy^2z+y^2z^2$.

[25] $a^2-2ab+b^2$, $3a^2+6ac+3c^2$, $a^2-2ax+x^2$, $9a^2-24ab+16b^2$, $25x^2+40xy+16y^2$, $3x^2y^2-36x^2y+108x^2$.

[26] $a^2+2a(b+c)+(b+c)^2$ or $a^2+2ab+b^2+2ac+2bc+c^2$, $b^2+2bc+c^2+2bd+2cd+d^2$, $x^2+2xy+y^2+2xz+2yz+z^2$, $a^2-2ab+b^2-2ac+2bc+c^2$, $b^2-2bc+c^2-2bd+2cd+d^2$, $x^2-2xy+y^2+2xz-2yz+z^2$.

[27] $4y^6+8xy^3+4x^2$, $-4x^2+8xy-y^2$, $-16a^2-32ab-32ac-16b^2-32bc-16c^2$, $12a^2-24ab+24ac+12b^2-24bc+12c^2$, $36x^2-96xy+64y^2+120xz-160yz+100z^2$, $2a^2-2ab-ac+\frac{1}{2}b^2+\frac{1}{2}bc+\frac{1}{8}c^2$.

[28] $x^2\pm 2xy+y^2$, $y^3\pm 3y^2z+3yz^2\pm z^3$, $a^3\pm 3a^2b+3ab^2\pm b^3$.

[29] a^4b^2, a^8b^6, $a^{2n}b^{10}$. [30] $8a^6$, $27a^9$, $-27a^{27}$, $-64a^{12}$.

[31] a^3x^3, $-a^{3n}b^6$, $a^{12}b^9c^3$, b^{3pq}. [32] $81a^{12}b^4c^{4n}$, $32a^{20}b^{15n}c^5$, $16a^8b^{4n}c^{8p}$.

[33] a^{mnpq}. [34] a^{mnpq}. [35] $6561p^{24}q^{18}s^8$, $1189\frac{6595}{16384}a^{7n}b^{7m}$.

[36] a^{2n}, $-a^{2n+1}$. [37] $16a^8-96a^6b+216a^4b^2-216a^2b^3+81b^4$, $243a^5c^5-810a^4c^4bd+1080a^3c^3b^2d^2-720a^2c^2b^3d^3+240acb^4d^4-32b^5d^5$, $64a^{18}-576a^{15}b^2c+2160a^{12}b^4c^2-4320a^9b^6c^3+4860a^6b^8c^4-2916a^3b^{10}c^5+729b^{12}c^6$. [38] $a^6-3a^4b^2-3a^4c^2+3a^2b^4+6a^2b^2c^2+3a^2c^4-b^6-3b^4c^2-3b^2c^4-c^6$, $343a^6-441a^5b+777a^4b^2-531a^3b^3+444a^2b^4-144ab^5+64b^6$.

[39] $a^3+3a^2b+3a^2c+3a^2d+3ab^2+6abc+6abd+3ac^2+6acd+3ad^2+b^3+3b^2c+3b^2d+3bc^2+6bcd+3bd^2+c^3+3c^2d+3cd^2+d^3$.

[40] $1\pm 11x+55x^2\pm 165x^3+330x^4\pm 462x^5+462x^6\pm 330x^7+165x^8\pm 55x^9+11x^{10}\pm x^{11}$. [41] $10a^3b^2$. [42] $1820a^{12}b^4$. [43] $-142506a^{25}b^5$.

[44] $495a^{16}b^8$. [45] $192192a^6b^6c^8d^8$. [46] $12870a^8b^8$.

[47] $10524937500a^{12}b^{12}c^6d^6$. [48] $24310a^9b^8-24310a^8b^9$.

Exercise XVIII.

[1] $\pm a$, $\pm b^2$, $\pm x^4$, $\pm y^5$, $\pm c^3$. [2] $-y^2$, $\pm x^2y^2$, $\pm a^3b^4$, $\pm x^9y^6$.
[3] $\pm(a+b)^2$, $\pm(a-b)^2$, $\pm x^3(x^2+y^2)$, $\pm(x+y)(x-y)$.
[4] a^2b^3, $-x^3y^4$, x^5y^6. [5] b^2, $\pm(a+b)^2$, a^pb^{2q}, $-x^ry^{2p}$.
[6] $\pm\frac{a^3}{b^4}$, $\pm\frac{ab^2}{c^2d^3}$, $\pm\frac{a^nb^p}{c^{2r}d^{3s}}$, $\pm\frac{(x+y)^2}{(y+x)^m}$. [7] $\frac{y^3}{x^5}$, $-\frac{xy^2}{z^3v^4}$, $\frac{a^2b^5c}{(a+f)h^3x^4}$.
[8] $2^4a^4b(a+b)^3$, $\frac{c}{a^2b^3d^2f^2}$.

Exercise XIX.

[1] 16, 64, 247, 763, 978, 7563. [2] 8276, 5083, 15367, 40093.
[3] 27943, 37695, 534762. [4] 203975, 6950078, 3476905.
[5] 2·23606, 3·60555, 4·69041, 9·79795.

EXERCISE XX.

[1] $a+b$. [2] $a-b$. [3] $a-\frac{b}{2}$. [4] $x+1$. [5] f^3+3x^4.
[6] $\frac{3a^4}{2}+\frac{2m^3}{3}$. [7] $\frac{5}{2}ab-\frac{1}{3}c^2$. [8] $x^2-\frac{1}{2}ax$. [9] a^m+x^n. [10] a^m-2a^n.
[11] $\frac{a}{b}-\frac{2b}{3c}$. [12] $a+b+c$. [13] $3x-5a-\frac{a^2}{2}$. [14] $2x^2+2ax+4b^2$.
[15] $3a-b+5c+d$. [16] $\frac{3}{2}+2x-7x^2$. [17] $3x^2-\frac{ax}{2}+5x$.
[18] $\frac{2}{3}ax^2-bxz+2abz^2$. [19] $\frac{a-b}{x^2+2a}$. [20] $\frac{ax+b^2x^2}{a^m+x^n}$.

EXERCISE XXI.

[1] 23, 96, 74, 135, 223. [2] 106, 258, 368, 343.
[3] 401, 200, 420, 683. [4] 698, 1854, 1936.
[5] 4820, 4865, 2667. [6] 4835, 2009.

[1] $a+b$. [2] $a-b$. [3] $x+2$. [4] $2a-7x$. [5] x^2-2cx.
[6] a^m-2ax^n. [7] $2-x^{3n-1}$. [8] $\frac{ac}{b}x^2-\frac{b}{c}x$. [9] $b+\frac{a^2}{2c^2x^2}$.
[10] $a+b+c$. [11] $3z^2-2az+a^2$. [12] $\frac{ay}{b^2c}+\frac{c^3y^2}{d}-a$.

EXERCISE XXII.

[1] $4x^2+4xy+4y^2$. [2] x^2y+xy^2. [3] $3a-3b$. [4] $2xy+3y^2$.
[5] $16bx^2-25by^2$. [6] $5a-5b$. [7] $x-y$. [8] $x-y$.
[9] $3x-3y$. [10] $\frac{x+y}{x^2+xy+y^2}$. [11] $\frac{x-y}{x+y}$. [12] $\frac{x+y}{3x-3y}$.

EXERCISE XXIII.

[1] $3x^2+3y^2$. [2] $x-2y$. [3] $x+4$. [4] $2a^3+ab^2-3b^3$.
[5] $3a^2+5b$. [6] $p-q$. [7] $2a-3b$. [8] $5x^3-3xy^2+2y^3$.
[9] $2x^3-4x^2+x-1$. [10] $2x-7$. [11] $9a^2x^3(x-1)$.
[12] $2a^2b(a^2-ab+2b^2)$.

EXERCISE XXIV.

[1] $360ab^2c^2x^3y^3z$. [2] $36a^2b^3c(x+y)^2(x-y)=36a^2b^3c(x^2-y^2)(x+y)$.
[3] $108ax^2y^2z^2(x+y)(x-y)(x^2+xy+y^2)$.
[4] $3xy(x+3y)(x+5y)=3x^3y+24x^2y^2+45xy^3$.
[5] $2ab(a+2b)(a+3b)=2a^3b+10a^2b^2+12ab^3$.
[6] $60a^2b^2cd(a+3b)(a-b)(a+b)^3$.
[7] $-120m^2(m-n)(m+n)^2(3m+2n)(m+1)(m-1)$.
[8] $(3x^2+2)(2x-3)(x+1)^2(x-2)(5x-3)(x-1)$.

EXERCISE XXV.

[1] $\frac{4}{5}\times\frac{a^2(x+y)}{x(x^2+xy+y^2)}$. [2] $\frac{a(a+1)}{3(a+2)}=\frac{1}{3}a\times\frac{a+1}{a+2}$. [3] $\frac{5}{6}\times\frac{x-y}{x^2-xy+y^2}$.

[4] $1\frac{4}{7}\times\frac{x-7}{x+6}$. [5] $\frac{5a+2}{7a^2-1}$. [6] $\frac{x+4}{x-9}$.

[7] $\frac{b+2c}{a^2-2b^2}$. [8] $\frac{x-8}{x+48}$.

EXERCISE XXVI.

[1] $\frac{(a-b)^2}{a^2-b^2}$. [2] $\frac{cd-c^2}{d^2-cd}$. [3] $\frac{2x^2+7x-9}{4x^2-81}$.

EXERCISE XXVII.

[1] $\frac{1}{2}(3a-9b)-\frac{b^2}{2a}$. [2] $1-\frac{10(a-1)}{a^2+8a-7}$. [3] $2x^2-4x+3-\frac{1}{x^2-x+3}$.

[4] $-5\frac{2}{3}a+\frac{b}{c}-\frac{c}{3a}+\frac{x^4}{3ac^3}$.

EXERCISE XXVIII.

[1] $\frac{a^2+b^2}{a}$. [2] $\frac{ac}{b}$. [3] $\frac{a(a-3b)}{a-b}$. [4] $\frac{6x^3-11x^2+4x-5}{3x^2-x+2}$.

[5] $\frac{x^4-2x^3-2x^2+4x-2}{x^2-3}$. [6] $\frac{3b^3}{a+b}$. [7] $\frac{10x^2y(x^2+2y^2)}{x^2+y^2}$.

[8] $\frac{10(p^4-q^4)}{2p^2-3q^2}$.

EXERCISE XXIX.

[1] $\frac{2ab-1}{3ab+1}$. [2] $\frac{a^3-b^3}{-2ab(a+b)}=\frac{b^3-a^3}{2ab(a+b)}$. [3] $\frac{1}{3a}\times\frac{3a-2b}{3a+2b}$.

[4] $\frac{x^2}{y}-2x=\frac{x(x-2y)}{y}$. [5] $\frac{2x^3-8x^2+11x-15}{2x^3-9x^2+9x-15}=1+\frac{x(x+2)}{2x^3-9x^2+9x-15}$.

[6] $\frac{2(a^2-b^2)}{a+4b}$. [7] $\frac{7a^3-9ab^2+2b^3}{9a^3-9ab^2-4b^3}$. [8] $\frac{(x+y)^2}{(x-y)^2}$.

[9] $\frac{(a-b)(a^2-2b^2)}{(a+b)(a^2+2b^2)}=\frac{a^3-a^2b-2ab^2+2b^3}{a^3+a^2b+2ab^2+2b^3}=1-\frac{2a^2b-4ab^2}{a^3+a^2b+2ab^2+2b^3}=1-$
$\frac{2ab(a-2b)}{(a+b)(a^2+2b^2)}$.

EXERCISE XXX.

[1] $\frac{40yz}{120x^3y^2z^2}$, $\frac{20xyz^2}{120x^3y^2z^2}$, $\frac{15x^3z}{120x^3y^2z^2}$, and $\frac{8x^3y}{120x^3y^2z^2}$.

[2] $\frac{a^2xy^2(x^2-y^2)}{60x^2y^3(x^2-y^2)}$, $\frac{35a^2by^2(x^2-y^2)}{60x^2y^3(x^2-y^2)}$, $\frac{68b^3x(x^2-y^2)}{60x^2y^3(x^2-y^2)}$, $\frac{480bx^2y^3(x+y)}{60x^2y^3(x^2-y^2)}$, and
$\frac{1020ax^2y^3(x-y)}{60x^2y^3(x^2-y^2)}$. [3] $\frac{a^2(a^2-b^2)}{(a^2-b^2)^2}$, $\frac{b^2(a-b)^2}{(a^2-b^2)^2}$, and $\frac{ab(a+b)^2}{(a^2-b^2)^2}$.

[4] $\frac{x^2+x-6}{x^4+x^3-10x^2-4x+24}$, $\frac{2(x^2+5x+6)}{x^4+x^3-10x^2-4x+24}$, and $\frac{3(x^2-4)}{x^4+x^3-10x^2-4x+24}$.

[5] $\frac{3(a-b)(x^3+x^2-x-1)}{3x^4-6x^2+3}$, $\frac{(a+b)(x^3+x^2-x-1)}{3x^4-6x^2+3}$, $\frac{3ab(x+1)^2}{3x^4-6x^2+3}$, $\frac{3a^2(x-1)^2}{3x^4-6x^2+3}$, and $\frac{b^2(x+1)^2}{3x^4-6x^2+3}$. [6] $\frac{(2a-b)^2}{4a^4-4a^3b+a^2b^2-2ab^3+b^4}$ and $\frac{(a-b)^2}{4a^4-4a^3b+a^2b^2-2ab^3+b^4}$.

EXERCISE XXXI.

[1] $\frac{x+y}{a}$. [2] $\frac{x-y+z-v-w}{a}$. [3] $\frac{a^2+2ab+b^2}{d}=\frac{(a+b)^2}{d}$. [4] $2x-3$.

[5] $\frac{(m-n)^2}{p-q}$. [6] $2a$. [7] $-ab\times\frac{(2a-b)^2}{a^2+2b^2}$. [8] $a\times\frac{x+y}{xy}$.

[9] $a\times\frac{y-x}{xy}$. [10] $\frac{ad+bc}{bd}$. [11] $\frac{bc+ac+ab}{abc}$.

[12] $\frac{30a^2qrs+20b^2prs-15c^2pqs+12d^2pqr}{60pqrs}$.

[13] $\frac{80xy^2p^2r^2+72x^2y^2zpr-100y^3z^3q-75xp^3qr^3}{120p^3qr^2}$. [14] $\frac{a^2+b^2}{a^2-b^2}=1+\frac{2b^2}{a^2-b^2}$.

[15] $\frac{(a+b)^2}{ab(a-b)}$. [16] $\frac{y}{x+y}$. [17] $\frac{5}{(4x-9)(3x-8)}$. [18] $-\frac{5x^2-8x+5}{x^2(x-1)}$.

[19] $\frac{5xy}{(3x-2y)(2x-3y)}$. [20] $\frac{bcd-acd+abd-abc}{abcd}=\frac{cd(b-a)+ab(d-c)}{abcd}$.

[21] $\frac{(a+b+c)x^2-2(ab+ac+bc)x+3abc}{x^3+(a+b+c)x^2+(ab+ac+bc)x-abc}$. [22] $\frac{6x^3+31x^2-66x+99}{x^2(x-3)^2}$.

[23] $\frac{791x^2-773x+192}{(7x-3)^3}$. [24] $\frac{16x^2+13xy+3y^2}{36(x+y)(x-y)^2}$. [25] $\frac{b^2}{a+b}$.

[26] $\frac{32x^3+37x^2-65x-60}{8x^4-6x^3-17x^2+6x+9}$. [27] $\frac{2b(10a^2+3b^2)}{5(a^2-b^2)}=\frac{2}{5}b\times\frac{10a^2+3b^2}{a^2-b^2}$.

[28] $2m+\frac{2n^4}{(m+n)(m^2+n^2)}=\frac{2(m^5-n^5)}{m^4-n^4}$. [29] $5(a-b)^2$.

[30] $12(a-b)-\frac{b(4a^2+13ab-9b^2)}{36a(a-b)}=\frac{432a^3-868a^2b+419ab^2+9b^3}{36a(a-b)}$.

EXERCISE XXXII.

[1] $\frac{ac}{bd}$. [2] $\frac{4ac}{5by}$. [3] $-\frac{15bx^2z}{4ay^2v}$. [4] $\frac{6p^3}{55qy}$. [5] $\frac{21a^2b^2}{x^2-8x+15}$.

[6] $\frac{x-a}{x+a}$. [7] $\frac{ab}{x^2-(a+b)x+ab}$. [8] $\frac{14ac^2(c-d)^2}{15b^2(a+b)(c+d)}$.

[9] $\frac{2a(a^3+b^3)}{3b(a-b)}$. [10] $\frac{a^2+2ab+b^2}{ab}$.

[11] $\frac{144a^4+31a^2b^2-14b^4}{12a^2}=12a^2+2\tfrac{7}{12}b^2-\frac{7b^4}{6a^2}$.

[12] $\frac{m^4n(p+q)}{pq(p-q)(p^2+q^2)}=\frac{m^4n(p+q)}{pq(p^3-p^2q+pq^2-q^3)}$. [13] a^3-b^3.

[14] $-\frac{(m+n)^2(3m^4+5n^4)}{m+2n}=-3m^5-3m^4n+9m^3n^2-18m^2n^3+41mn^4-72n^5+\frac{149n^6}{m+2n}$. [15] $\frac{9}{35}(x-y)$. [16] $\frac{9x^{n-2}y^{m-2}(a+b)}{a-b}$.

[17] $\frac{a^3+a^2b+ab^2+b^3}{a^3b^3}=\frac{1}{b^3}+\frac{1}{a^3}+\frac{1}{ab}\left\{\frac{1}{a}+\frac{1}{b}\right\}$.

[18] $\frac{(x-1)(y-1)(z-1)}{xyz}=\frac{xyz-(xy+xz+yz)+(x+y+z)-1}{xyz}$.

EXERCISE XXXIII.

[1] $\frac{ay}{bx}$. [2] $\frac{3bx^3}{5xy^2}$. [3] $\frac{45a^4}{b}$. [4] $\frac{2x}{45x^3}$. [5] $\frac{3(x+y)}{2x^2y^2}$. [6] $\frac{a(a+y)}{a-b}$.
[7] $\frac{a^2-ab+b^2}{3(a-b)}$. [8] $\frac{(3x+y)(x+y)}{(x+3y)(x-y)}=\frac{3x^2+4xy+y^2}{x^2+2xy-3y^2}$. [9] $\frac{x}{y}$.
[10] $3\times\frac{5a+7y}{5a-7y}$. [11] $\frac{19}{(19x-y)(x-y)}$. [12] $\frac{15(x^2+xy+y^2)}{7xy}$.
[13] $\frac{a^2p}{b(aq+bq)}$. [14] $\frac{5x^{p-3}y^{q-2}}{4(a+b)}$. [15] $\frac{p^2+pq+3q^2}{p^2-2pq-3q^2}$.
[16] 1. [17] $\frac{apq}{2c}$. [18] $\frac{ab+c-d}{b}$. [19] $\frac{4xy}{(x^2-y^2)^2}$. [20] 1.

EXERCISE XXXIV.

[1] 1200. [2] $\frac{1075}{1296}$. [3] 2304. [4] $\frac{183}{384}$. [5] $-920\frac{28}{39}$.
[6] $12x^2+18xy-11y^2$. [7] $22a^4+4a^3b-16a^2b^2-4ab^3+2b^4+12d^3\sqrt{xy}+(3d^3-2e^3)\sqrt{ab}$. [8] $10x^3-16x^2y+3xy^2-3y^3+7p^2q-6h^2\sqrt{xy}-4m^3$. [9] $x^2+3xy-10y^2+(3a-2b)x-(a-b)y-ab(5c-a)$.
[10] $2a^4-a^3b-2a^2b^2-9ab^3+7b^4+3x^2y^2$. [11] $\frac{1}{12}(5a-36b-37x)$.
[12] $-(7x^3+9x^2y-14xy^2-6y^3)$. [13] $\frac{3}{4}x^6y^2z^2(a+x)^{10}(a+y)^5$.
[14] $-(\frac{7}{10}a^7y^2-\frac{2}{5}a^6y^3+\frac{14}{9}a^5y^4-\frac{28}{15}a^4y^5+\frac{14}{45}a^3y^6+\frac{7}{9}a^2y^7)$.
[15] $21x^7-14x^6y+22x^5y^2+9x^4y^3+10x^3y^4-24x^2y^5+10xy^6+8y^7$.
[16] $x^4+x^2y^2+y^4$. [17] $x^6+x^4y^2-x^2y^4-y^6$.
[18] $5x^4+3\frac{1}{2}ax^3-8\frac{1}{12}a^2x^2+\frac{5}{6}a^3x+1\frac{1}{6}a^4$. [19] $4a^3-5a^2b-3ac^2+6c^3$.
[20] $A=(a^2+b^2)(x^2+y^2)$. $C=(a^2+b^2+c^2)(x^2+y^2+z^2)$.
$B=(a^2+b^2+c^2)(x^2+y^2)$. $D=(a^2+b^2+c^2+d^2)(x^2+y^2+z^2+v^2)$.
[21] $\frac{1}{12}x(249x+242)$. [22] $10a^2by-10b^3y-16ax^3y+20ax^2y^4-20ay^6$.
[23] $a^{2p}-b^{2q}-c^{2r}+2b^qc^r$. [24] $ab^2+2abc-b^2c+c^3$.
[25] $x^p+y^q+2z^r$. [26] $\frac{2}{3}a^3-\frac{4}{5}a^2b+\frac{1}{2}b^3$. [27] $\frac{3}{4}a+\frac{1}{4}b$.
[28] $x^4+x^3+x^2+x+1$. [29] $1+x+x^2+x^3+x^4+\dots+x^{m-1}$.
[30] $(3ab+c)(a+b)$. [31] $(x^2+xy+y^2)(x^2-xy+y^2)$ and $(x+y)(x-y)(x^2+y^2)^2$ or $(x^3+x^2y+xy^2+y^3)(x^3-x^2y+xy^2-y^3)$.
[32] $\frac{a^4-2a^3+a^2+0{\cdot}15a+0{\cdot}02}{a(a-2)}=a^2+1+\frac{2{\cdot}15a+0{\cdot}02}{a(a-2)}$. [33] $12(x-1)$.
[34] $a-y$. [35] $\frac{1+x+x^2}{1-x-x^4+x^5}$. [36] $\frac{a+b+c}{a-b-c}$ and $\frac{a-2b}{a+b-c}$.
[37] $\frac{hkm(adf+bcf+bde)}{bdf(gkm+him+hkl)}$ and $\frac{a^2+2ab-b^2}{a^2+b^2}$. [38] $\frac{a+v}{a}$ and $\frac{a+2v}{a}$.
[39] $4a$. [40] $-\frac{a^2-(b+c+d)d+b(b+c)}{a(a^2-b^2)}$. [41] $\frac{(y-1)x^2+y(y+1)x-y^3}{x(x^3-y^3)}$.
[42] 1st product $=a-b$, the 2d $=x^2-xy+y^2$.
[43] $\frac{(a+b)x-c}{x}=a+b-\frac{c}{x}$ and $\frac{ab+x-y}{b}=a+\frac{x-y}{b}$. [44] a^3-b^3.

[5] $\frac{3(a-b)(z^3+z^2-z-1)}{3z^4-6z^2+3}$, $\frac{(a+b)(z^3+z^2-z-1)}{3z^4-6z^2+3}$, $\frac{3ab(z+1)^2}{3z^4-6z^2+3}$, $\frac{3a^2(z-1)^2}{3z^4-6z^2+3}$, and $\frac{b^2(z+1)^2}{3z^4-6z^2+3}$. [6] $\frac{(2a-b)^2}{4a^4-4a^3b+a^2b^2-2ab^3+b^4}$ and $\frac{(a-b)^2}{4a^4-4a^3b+a^2b^2-2ab^3+b^4}$.

EXERCISE XXXI.

[1] $\frac{x+y}{a}$. [2] $\frac{x-y+z-v-w}{a}$. [3] $\frac{a^2+2ab+b^2}{d}=\frac{(a+b)^2}{d}$. [4] $2x-3$.

[5] $\frac{(m-n)^2}{p-q}$. [6] $2a$. [7] $-ab\times\frac{(2a-b)^2}{a^2+2b^2}$. [8] $a\times\frac{x+y}{xy}$.

[9] $a\times\frac{y-z}{xy}$. [10] $\frac{ad+bc}{bd}$. [11] $\frac{bc+ac+ab}{abc}$.

[12] $\frac{30a^2qrs+20b^2prs-15c^2pqs+12d^2pqr}{60pqrs}$.

[13] $\frac{80xy^2p^2r^2+72x^2y^2zpr-100y^3z^3q-75xp^3qr^3}{120p^3qr^2}$. [14] $\frac{a^2+b^2}{a^2-b^2}=1+\frac{2b^2}{a^2-b^2}$.

[15] $\frac{(a+b)^2}{ab(a-b)}$. [16] $\frac{y}{x+y}$. [17] $\frac{5}{(4x-9)(3x-8)}$. [18] $-\frac{5x^2-8x+5}{x^2(x-1)}$.

[19] $\frac{5xy}{(3x-2y)(2x-3y)}$. [20] $\frac{bcd-acd+abd-abc}{abcd}=\frac{cd(b-a)+ab(d-c)}{abcd}$.

[21] $\frac{(a+b+c)x^2-2(ab+ac+bc)x+3abc}{x^3+(a+b+c)x^2+(ab+ac+bc)x-abc}$. [22] $\frac{6x^3+31x^2-66x+99}{x^2(x-3)^2}$.

[23] $\frac{791x^2-773x+192}{(7x-3)^3}$. [24] $\frac{16x^2+13xy+3y^2}{36(x+y)(x-y)^2}$. [25] $\frac{b^2}{a+b}$.

[26] $\frac{32x^3+37x^2-65x-60}{8x^4-6x^3-17x^2+6x+9}$. [27] $\frac{2b(10a^2+3b^2)}{5(a^2-b^2)}=\frac{2}{5}b\times\frac{10a^2+3b^2}{a^2-b^2}$.

[28] $2m+\frac{2n^4}{(m+n)(m^2+n^2)}=\frac{2(m^5-n^5)}{m^4-n^4}$. [29] $5(a-b)^2$.

[30] $12(a-b)-\frac{b(4a^2+13ab-9b^2)}{36a(a-b)}=\frac{432a^3-868a^2b+419ab^2+9b^3}{36a(a-b)}$.

EXERCISE XXXII.

[1] $\frac{ac}{bd}$. [2] $\frac{4ac}{5by}$. [3] $-\frac{15bx^2z}{4ay^2v}$. [4] $\frac{6p^3}{55qy}$. [5] $\frac{21a^2b^2}{x^2-8x+15}$.

[6] $\frac{x-a}{x+a}$. [7] $\frac{ab}{x^2-(a+b)x+ab}$. [8] $\frac{14ac^2(c-d)^2}{15b^2(a+b)(c+d)}$.

[9] $\frac{2a(a^3+b^3)}{3b(a-b)}$. [10] $\frac{a^2+2ab+b^2}{ab}$.

[11] $\frac{144a^4+31a^2b^2-14b^4}{12a^2}=12a^2+2\tfrac{7}{12}b^2-\frac{7b^4}{6a^2}$.

[12] $\frac{m^4n(p+q)}{pq(p-q)(p^2+q^2)}=\frac{m^4n(p+q)}{pq(p^3-p^2q+pq^2-q^3)}$. [13] a^3-b^3.

[14] $-\frac{(m+n)^2(3m^4+5n^4)}{m+2n}=-3m^5-3m^4n+9m^3n^2-18m^2n^3+41mn^4-72n^5+\frac{149n^6}{m+2n}$. [15] $\frac{9}{35}(x-y)$. [16] $\frac{9x^{n-2}y^{m-2}(a+b)}{a-b}$.

[17] $\frac{a^3+a^2b+ab^2+b^3}{a^3b^3}=\frac{1}{b^3}+\frac{1}{a^3}+\frac{1}{ab}\left\{\frac{1}{a}+\frac{1}{b}\right\}$.

[18] $\frac{(x-1)(y-1)(z-1)}{xyz}=\frac{xyz-(xy+xz+yz)+(x+y+z)-1}{xyz}$.

EXERCISE XXXIII.

[1] $\frac{ay}{bx}$. [2] $\frac{3bx^3}{5xy^2}$. [3] $\frac{45a^4}{b}$. [4] $\frac{2x}{45x^3}$. [5] $\frac{3(x+y)}{2x^2y^2}$. [6] $\frac{a(a+y)}{a-b}$.
[7] $\frac{a^2-ab+b^2}{3(a-b)}$. [8] $\frac{(3x+y)(x+y)}{(x+3y)(x-y)}=\frac{3x^2+4xy+y^2}{x^2+2xy-3y^2}$. [9] $\frac{x}{y}$.
[10] $3\times\frac{5a+7y}{5a-7y}$. [11] $\frac{19}{(19x-y)(x-y)}$. [12] $\frac{15(x^2+xy+y^2)}{7xy}$.
[13] $\frac{a^2p}{b(aq+bq)}$. [14] $\frac{5x^{p-3}y^{q-2}}{4(a+b)}$. [15] $\frac{p^2+pq+3q^2}{p^2-2pq-3q^2}$.
[16] 1. [17] $\frac{apq}{2c}$. [18] $\frac{ab+c-d}{b}$. [19] $\frac{4xy}{(x^2-y^2)^2}$. [20] 1.

EXERCISE XXXIV.

[1] 1200. [2] $\frac{1075}{1296}$. [3] 2304. [4] $\frac{183}{384}$. [5] $-920\frac{23}{39}$.
[6] $12x^2+18xy-11y^2$. [7] $22a^4+4a^3b-16a^2b^2-4ab^3+2b^4+12d^3\sqrt{xy}+(3d^3-2e^3)\sqrt{ab}$. [8] $10x^3-16x^2y+3xy^2-3y^3+7p^2q-6h^2\sqrt{xy}-4m^3$. [9] $x^2+3xy-10y^2+(3a-2b)x-(a-b)y-ab(5c-a)$.
[10] $2a^4-a^3b-2a^2b^2-9ab^3+7b^4+3x^2y^2$. [11] $\frac{1}{12}(5a-36b-37x)$.
[12] $-(7x^3+9x^2y-14xy^2-6y^3)$. [13] $\frac{3}{4}x^6y^2z^2(a+x)^{10}(a+y)^5$.
[14] $-(\frac{7}{10}a^7y^2-\frac{2}{5}a^6y^3+\frac{14}{9}a^5y^4-\frac{28}{75}a^4y^5+\frac{14}{45}a^3y^6+\frac{7}{9}a^2y^7)$.
[15] $21x^7-14x^6y+22x^5y^2+9x^4y^3+10x^3y^4-24x^2y^5+10xy^6+8y^7$.
[16] $x^4+x^2y^2+y^4$. [17] $x^6+x^4y^2-x^2y^4-y^6$.
[18] $5x^4+3\frac{1}{2}ax^3-8\frac{1}{12}a^2x^2+\frac{5}{6}a^3x+1\frac{1}{6}a^4$. [19] $4a^3-5a^2b-3ac^2+6c^3$.
[20] $A=(a^2+b^2)(x^2+y^2)$. $C=(a^2+b^2+c^2)(x^2+y^2+z^2)$.
$B=(a^2+b^2+c^2)(x^2+y^2)$. $D=(a^2+b^2+c^2+d^2)(x^2+y^2+z^2+v^2)$.
[21] $\frac{1}{12}x(249x+242)$. [22] $10a^2by-10b^3y-16ax^3y+20ax^2y^4-20ay^6$.
[23] $a^{2p}-b^{2q}-c^{2r}+2b^qc^r$. [24] $ab^2+2abc-b^2c+c^3$.
[25] $x^p+y^q+2z^r$. [26] $\frac{2}{3}a^3-\frac{4}{5}a^2b+\frac{1}{2}b^3$. [27] $\frac{3}{7}a+\frac{1}{4}b$.
[28] $x^4+x^3+x^2+x+1$. [29] $1+x+x^2+x^3+x^4+\;\ldots\;+x^{m-1}$.
[30] $(3ab+c)(a+b)$. [31] $(x^2+xy+y^2)(x^2-xy+y^2)$ and $(x+y)(x-y)(x^2+y^2)^2$ or $(x^3+x^2y+xy^2+y^3)(x^3-x^2y+xy^2-y^3)$.
[32] $\frac{a^4-2a^3+a^2+0{\cdot}15a+0{\cdot}02}{a(a-2)}=a^2+1+\frac{2{\cdot}15a+0{\cdot}02}{a(a-2)}$. [33] $12(x-1)$.
[34] $a-y$. [35] $\frac{1+x+x^2}{1-x-x^4+x^5}$. [36] $\frac{a+b+c}{a-b-c}$ and $\frac{a-2b}{a+b-c}$.
[37] $\frac{hkm(adf+bcf+bde)}{bdf(gkm+him+hkl)}$ and $\frac{a^2+2ab-b^2}{a^2+b^2}$. [38] $\frac{a+v}{a}$ and $\frac{a+2v}{a}$.
[39] $4a$. [40] $-\frac{a^2-(b+c+d)d+b(b+c)}{a(a^2-b^2)}$. [41] $\frac{(y-1)x^2+y(y+1)x-y^3}{x(x^3-y^3)}$.
[42] 1st product $=a-b$, the 2d $=x^2-xy+y^2$.
[43] $\frac{(a+b)x-c}{x}=a+b-\frac{c}{x}$ and $\frac{ab+x-y}{b}=a+\frac{x-y}{b}$. [44] a^3-b^3.

[5] $\frac{3(a-b)(z^3+z^2-z-1)}{3z^4-6z^2+3}$, $\frac{(a+b)(z^3+z^2-z-1)}{3z^4-6z^2+3}$, $\frac{3ab(z+1)^2}{3z^4-6z^2+3}$, $\frac{3a^2(z-1)^2}{3z^4-6z^2+3}$, and $\frac{b^2(z+1)^2}{3z^4-6z^2+3}$. [6] $\frac{(2a-b)^2}{4a^4-4a^3b+a^2b^2-2ab^3+b^4}$ and $\frac{(a-b)^2}{4a^4-4a^3b+a^2b^2-2ab^3+b^4}$.

EXERCISE XXXI.

[1] $\frac{x+y}{a}$. [2] $\frac{x-y+z-v-w}{a}$. [3] $\frac{a^2+2ab+b^2}{d}=\frac{(a+b)^2}{d}$. [4] $2x-3$.

[5] $\frac{(m-n)^2}{p-q}$. [6] $2a$. [7] $-ab\times\frac{(2a-b)^2}{a^2+2b^2}$. [8] $a\times\frac{x+y}{xy}$.

[9] $a\times\frac{y-z}{xy}$. [10] $\frac{ad+bc}{bd}$. [11] $\frac{bc+ac+ab}{abc}$.

[12] $\frac{30a^2qrs+20b^2prs-15c^2pqs+12d^2pqr}{60pqrs}$.

[13] $\frac{80xy^2p^2r^2+72x^2y^2zpr-100y^3z^3q-75xp^3qr^3}{120p^3qr^2}$. [14] $\frac{a^2+b^2}{a^2-b^2}=1+\frac{2b^2}{a^2-b^2}$.

[15] $\frac{(a+b)^2}{ab(a-b)}$. [16] $\frac{y}{x+y}$. [17] $\frac{5}{(4x-9)(3x-8)}$. [18] $-\frac{5x^2-8x+5}{x^2(x-1)}$.

[19] $\frac{5xy}{(3x-2y)(2x-3y)}$. [20] $\frac{bcd-acd+abd-abc}{abcd}=\frac{cd(b-a)+ab(d-c)}{abcd}$.

[21] $\frac{(a+b+c)x^2-2(ab+ac+bc)x+3abc}{x^3+(a+b+c)x^2+(ab+ac+bc)x-abc}$. [22] $\frac{6x^3+31x^2-66x+99}{x^2(x-3)^2}$.

[23] $\frac{791x^2-773x+192}{(7x-3)^3}$. [24] $\frac{16x^2+13xy+3y^2}{36(x+y)(x-y)^2}$. [25] $\frac{b^2}{a+b}$.

[26] $\frac{32x^3+37x^2-65x-60}{8x^4-6x^3-17x^2+6x+9}$. [27] $\frac{2b(10a^2+3b^2)}{5(a^2-b^2)}=\frac{2}{5}b\times\frac{10a^2+3b^2}{a^2-b^2}$.

[28] $2m+\frac{2n^4}{(m+n)(m^2+n^2)}=\frac{2(m^5-n^5)}{m^4-n^4}$. [29] $5(a-b)^2$.

[30] $12(a-b)-\frac{b(4a^2+13ab-9b^2)}{36a(a-b)}=\frac{432a^3-868a^2b+419ab^2+9b^3}{36a(a-b)}$.

EXERCISE XXXII.

[1] $\frac{ac}{bd}$. [2] $\frac{4ac}{5by}$. [3] $-\frac{15bx^2z}{4ay^2v}$. [4] $\frac{6p^3}{55qy}$. [5] $\frac{21a^2b^2}{x^2-8x+15}$.

[6] $\frac{x-a}{x+a}$. [7] $\frac{ab}{x^2-(a+b)x+ab}$. [8] $\frac{14ac^2(c-d)^2}{15b^2(a+b)(c+d)}$.

[9] $\frac{2a(a^3+b^3)}{3b(a-b)}$. [10] $\frac{a^2+2ab+b^2}{ab}$.

[11] $\frac{144a^4+31a^2b^2-14b^4}{12a^2}=12a^2+2\frac{7}{12}b^2-\frac{7b^4}{6a^2}$.

[12] $\frac{m^4n(p+q)}{pq(p-q)(p^2+q^2)}=\frac{m^4n(p+q)}{pq(p^3-p^2q+pq^2-q^3)}$. [13] a^3-b^3.

[14] $-\frac{(m+n)^2(3m^4+5n^4)}{m+2n}=-3m^5-3m^4n+9m^3n^2-18m^2n^3+41mn^4-72n^5+\frac{149n^6}{m+2n}$. [15] $\frac{9}{35}(x-y)$. [16] $\frac{9x^{n-2}y^{m-2}(a+b)}{a-b}$.

[17] $\frac{a^3+a^2b+ab^2+b^3}{a^3b^3}=\frac{1}{b^3}+\frac{1}{a^3}+\frac{1}{ab}\left\{\frac{1}{a}+\frac{1}{b}\right\}$.

[18] $\frac{(x-1)(y-1)(z-1)}{xyz}=\frac{xyz-(xy+xz+yz)+(x+y+z)-1}{xyz}$.

EXERCISE XXXIII.

[1] $\frac{ay}{bx}$. [2] $\frac{3bx^3}{5xy^2}$. [3] $\frac{45a^4}{b}$. [4] $\frac{2x}{45x^3}$. [5] $\frac{3(x+y)}{2x^2y^2}$. [6] $\frac{a(a+y)}{a-b}$.

[7] $\frac{a^2-ab+b^2}{3(a-b)}$. [8] $\frac{(3x+y)(x+y)}{(x+3y)(x-y)}=\frac{3x^2+4xy+y^2}{x^2+2xy-3y^2}$. [9] $\frac{x}{y}$.

[10] $3\times\frac{5a+7y}{5a-7y}$. [11] $\frac{19}{(19x-y)(x-y)}$. [12] $\frac{15(x^2+xy+y^2)}{7xy}$.

[13] $\frac{a^2p}{b(aq+bq)}$. [14] $\frac{5x^{p-3}y^{q-2}}{4(a+b)}$. [15] $\frac{p^2+pq+3q^2}{p^2-2pq-3q^2}$.

[16] 1. [17] $\frac{apq}{2c}$. [18] $\frac{ab+c-d}{b}$. [19] $\frac{4xy}{(x^2-y^2)^2}$. [20] 1.

EXERCISE XXXIV.

[1] 1200. [2] $\frac{1075}{1296}$. [3] 2304. [4] $\frac{133}{384}$. [5] $-920\frac{28}{39}$.
[6] $12x^2+18xy-11y^2$. [7] $22a^4+4a^3b-16a^2b^2-4ab^3+2b^4+12d^3\sqrt{xy}+(3d^3-2e^3)\sqrt{ab}$. [8] $10x^3-16x^2y+3xy^2-3y^3+7p^2q-6h^2\sqrt{xy}-4m^3$. [9] $x^2+3xy-10y^2+(3a-2b)x-(a-b)y-ab(5c-a)$.
[10] $2a^4-a^3b-2a^2b^2-9ab^3+7b^4+3x^2y^2$. [11] $\frac{1}{12}(5a-36b-37x)$.
[12] $-(7x^3+9x^2y-14xy^2-6y^3)$. [13] $\frac{3}{4}x^6y^2z^2(a+x)^{10}(a+y)^5$.
[14] $-(\frac{7}{10}a^7y^2-\frac{2}{5}a^6y^3+\frac{14}{9}a^5y^4-\frac{28}{75}a^4y^5+\frac{14}{45}a^3y^6+\frac{7}{9}a^2y^7)$.
[15] $21x^7-14x^6y+22x^5y^2+9x^4y^3+10x^3y^4-24x^2y^5+10xy^6+8y^7$.
[16] $x^4+x^2y^2+y^4$. [17] $x^6+x^4y^2-x^2y^4-y^6$.
[18] $5x^4+3\frac{1}{2}ax^3-8\frac{1}{12}a^2x^2+\frac{5}{6}a^3x+1\frac{1}{6}a^4$. [19] $4a^3-5a^2b-3ac^2+6c^3$.
[20] $A=(a^2+b^2)(x^2+y^2)$. $C=(a^2+b^2+c^2)(x^2+y^2+z^2)$.
$B=(a^2+b^2+c^2)(x^2+y^2)$. $D=(a^2+b^2+c^2+d^2)(x^2+y^2+z^2+v^2)$.
[21] $\frac{1}{12}x(249x+242)$. [22] $10a^2by-10b^3y-16ax^3y+20ax^2y^4-20ay^6$.
[23] $a^{2p}-b^{2q}-c^{2r}+2b^qc^r$. [24] $ab^2+2abc-b^2c+c^3$.
[25] $x^p+y^q+2z^r$. [26] $\frac{2}{3}a^3-\frac{4}{5}a^2b+\frac{1}{2}b^3$. [27] $\frac{3}{7}a+\frac{1}{4}b$.
[28] $x^4+x^3+x^2+x+1$. [29] $1+x+x^2+x^3+x^4+\ \ldots\ +x^{m-1}$.
[30] $(3ab+c)(a+b)$. [31] $(x^2+xy+y^2)(x^2-xy+y^2)$ and $(x+y)(x-y)(x^2+y^2)^2$ or $(x^3+x^2y+xy^2+y^3)(x^3-x^2y+xy^2-y^3)$.
[32] $\frac{a^4-2a^3+a^2+0{\cdot}15a+0{\cdot}02}{a(a-2)}=a^2+1+\frac{2{\cdot}15a+0{\cdot}02}{a(a-2)}$. [33] $12(x-1)$.
[34] $a-y$. [35] $\frac{1+x+x^2}{1-x-x^4+x^5}$. [36] $\frac{a+b+c}{a-b-c}$ and $\frac{a-2b}{a+b-c}$.
[37] $\frac{hkm(adf+bcf+bde)}{bdf(gkm+him+hkl)}$ and $\frac{a^2+2ab-b^2}{a^2+b^2}$. [38] $\frac{a+v}{a}$ and $\frac{a+2v}{a}$.
[39] $4a$. [40] $-\frac{a^2-(b+c+d)d+b(b+c)}{a(a^2-b^2)}$. [41] $\frac{(y-1)x^2+y(y+1)x-y^3}{x(x^3-y^3)}$.
[42] 1st product $=a-b$, the 2d $=x^2-xy+y^2$.
[43] $\frac{(a+b)x-c}{x}=a+b-\frac{c}{x}$ and $\frac{ab+x-y}{b}=a+\frac{x-y}{b}$. [44] a^3-b^3.

[5] $\frac{3(a-b)(z^3+z^2-z-1)}{3z^4-6z^2+3}$, $\frac{(a+b)(z^3+z^2-z-1)}{3z^4-6z^2+3}$, $\frac{3ab(z+1)^2}{3z^4-6z^2+3}$, $\frac{3a^2(z-1)^2}{3z^4-6z^2+3}$, and $\frac{b^2(z+1)^2}{3z^4-6z^2+3}$. [6] $\frac{(2a-b)^2}{4a^4-4a^3b+a^2b^2-2ab^3+b^4}$ and $\frac{(a-b)^2}{4a^4-4a^3b+a^2b^2-2ab^3+b^4}$.

EXERCISE XXXI.

[1] $\frac{x+y}{a}$. [2] $\frac{x-y+z-v-w}{a}$. [3] $\frac{a^2+2ab+b^2}{d}=\frac{(a+b)^2}{d}$. [4] $2x-3$.

[5] $\frac{(m-n)^2}{p-q}$. [6] $2a$. [7] $-ab\times\frac{(2a-b)^2}{a^2+2b^2}$. [8] $a\times\frac{x+y}{xy}$.

[9] $a\times\frac{y-z}{xy}$. [10] $\frac{ad+bc}{bd}$. [11] $\frac{bc+ac+ab}{abc}$.

[12] $\frac{30a^2qrs+20b^2prs-15c^2pqs+12d^2pqr}{60pqrs}$.

[13] $\frac{80xy^2p^2r^3+72x^2y^2zpr-100y^3z^3q-75xp^3qr^3}{120p^3qr^2}$. [14] $\frac{a^2+b^2}{a^2-b^2}=1+\frac{2b^2}{a^2-b^2}$.

[15] $\frac{(a+b)^2}{ab(a-b)}$. [16] $\frac{y}{x+y}$. [17] $\frac{5}{(4x-9)(3x-8)}$. [18] $-\frac{5x^2-8x+5}{x^2(x-1)}$.

[19] $\frac{5xy}{(3x-2y)(2x-3y)}$. [20] $\frac{bcd-acd+abd-abc}{abcd}=\frac{cd(b-a)+ab(d-c)}{abcd}$.

[21] $\frac{(a+b+c)x^2-2(ab+ac+bc)x+3abc}{x^3+(a+b+c)x^2+(ab+ac+bc)x-abc}$. [22] $\frac{6x^3+31x^2-66x+99}{x^2(x-3)^2}$.

[23] $\frac{791x^2-773x+192}{(7x-3)^3}$. [24] $\frac{16x^2+13xy+3y^2}{36(x+y)(x-y)^2}$. [25] $\frac{b^2}{a+b}$.

[26] $\frac{32x^3+37x^2-65x-60}{8x^4-6x^3-17x^2+6x+9}$. [27] $\frac{2b(10a^2+3b^2)}{5(a^2-b^2)}=\frac{2}{5}b\times\frac{10a^2+3b^2}{a^2-b^2}$.

[28] $2m+\frac{2n^4}{(m+n)(m^2+n^2)}=\frac{2(m^5-n^5)}{m^4-n^4}$. [29] $5(a-b)^2$.

[30] $12(a-b)-\frac{b(4a^2+13ab-9b^2)}{36a(a-b)}=\frac{432a^3-868a^2b+419ab^2+9b^3}{36a(a-b)}$.

EXERCISE XXXII.

[1] $\frac{ac}{bd}$. [2] $\frac{4ac}{5by}$. [3] $-\frac{15bx^2z}{4ay^2v}$. [4] $\frac{6p^3}{55qy}$. [5] $\frac{21a^2b^2}{x^2-8x+15}$.

[6] $\frac{x-a}{x+a}$. [7] $\frac{ab}{x^2-(a+b)x+ab}$. [8] $\frac{14ac^2(c-d)^2}{15b^2(a+b)(c+d)}$.

[9] $\frac{2a(a^3+b^3)}{3b(a-b)}$. [10] $\frac{a^2+2ab+b^2}{ab}$.

[11] $\frac{144a^4+31a^2b^2-14b^4}{12a^2}=12a^2+2\frac{7}{12}b^2-\frac{7b^4}{6a^2}$.

[12] $\frac{m^4n(p+q)}{pq(p-q)(p^2+q^2)}=\frac{m^4n(p+q)}{pq(p^3-p^2q+pq^2-q^3)}$. [13] a^3-b^3.

[14] $-\frac{(m+n)^2(3m^4+5n^4)}{m+2n}=-3m^5-3m^4n+9m^3n^2-18m^2n^3+41mn^4-72n^5+\frac{149n^6}{m+2n}$. [15] $\frac{9}{35}(x-y)$. [16] $\frac{9x^{n-2}y^{m-2}(a+b)}{a-b}$.

[17] $\frac{a^3+a^2b+ab^2+b^3}{a^3b^3}=\frac{1}{b^3}+\frac{1}{a^3}+\frac{1}{ab}\left\{\frac{1}{a}+\frac{1}{b}\right\}$.

[18] $\frac{(x-1)(y-1)(z-1)}{xyz}=\frac{xyz-(xy+xz+yz)+(x+y+z)-1}{xyz}$.

EXERCISE XXXIII.

[1] $\frac{ay}{bx}$. [2] $\frac{3bx^3}{5xy^2}$. [3] $\frac{45a^4}{b}$. [4] $\frac{2x}{45x^3}$. [5] $\frac{3(x+y)}{2x^2y^2}$. [6] $\frac{a(a+y)}{a-b}$.

[7] $\frac{a^2-ab+b^2}{3(a-b)}$. [8] $\frac{(3x+y)(x+y)}{(x+3y)(x-y)}=\frac{3x^2+4xy+y^2}{x^2+2xy-3y^2}$. [9] $\frac{x}{y}$.

[10] $3\times\frac{5a+7y}{5a-7y}$. [11] $\frac{19}{(19x-y)(x-y)}$. [12] $\frac{15(x^2+xy+y^2)}{7xy}$.

[13] $\frac{a^2p}{b(aq+bq)}$. [14] $\frac{5x^{p-3}y^{q-2}}{4(a+b)}$. [15] $\frac{p^2+pq+3q^2}{p^2-2pq-3q^2}$.

[16] 1. [17] $\frac{apq}{2c}$. [18] $\frac{ab+c-d}{b}$. [19] $\frac{4xy}{(x^2-y^2)^2}$. [20] 1.

EXERCISE XXXIV.

[1] 1200. [2] $\frac{1075}{1296}$. [3] 2304. [4] $\frac{183}{384}$. [5] $-920\frac{23}{39}$.

[6] $12x^2+18xy-11y^2$. [7] $22a^4+4a^3b-16a^2b^2-4ab^3+2b^4+12d^3\sqrt{xy}+(3d^3-2e^3)\sqrt{ab}$. [8] $10x^3-16x^2y+3xy^2-3y^3+7p^2q-6h^2\sqrt{xy}-4m^3$. [9] $x^2+3xy-10y^2+(3a-2b)x-(a-b)y-ab(5c-a)$.

[10] $2a^4-a^3b-2a^2b^2-9ab^3+7b^4+3x^2y^2$. [11] $\frac{1}{12}(5a-36b-37x)$.

[12] $-(7x^3+9x^2y-14xy^2-6y^3)$. [13] $\frac{3}{4}x^6y^2z^2(a+x)^{10}(a+y)^5$.

[14] $-(\frac{7}{10}a^7y^2-\frac{2}{5}a^6y^3+\frac{14}{9}a^5y^4-\frac{28}{75}a^4y^5+\frac{14}{45}a^3y^6+\frac{7}{9}a^2y^7)$.

[15] $21x^7-14x^6y+22x^5y^2+9x^4y^3+10x^3y^4-24x^2y^5+10xy^6+8y^7$.

[16] $x^4+x^2y^2+y^4$. [17] $x^6+x^4y^2-x^2y^4-y^6$.

[18] $5x^4+3\frac{1}{2}ax^3-8\frac{1}{12}a^2x^2+\frac{5}{6}a^3x+1\frac{1}{6}a^4$. [19] $4a^3-5a^2b-3ac^2+6c^3$.

[20] $A=(a^2+b^2)(x^2+y^2)$. $C=(a^2+b^2+c^2)(x^2+y^2+z^2)$.
$B=(a^2+b^2+c^2)(x^2+y^2)$. $D=(a^2+b^2+c^2+d^2)(x^2+y^2+z^2+v^2)$.

[21] $\frac{1}{12}x(249x+242)$. [22] $10a^2by-10b^3y-16ax^3y+20ax^2y^4-20ay^6$.

[23] $a^{2p}-b^{2q}-c^{2r}+2b^qc^r$. [24] $ab^2+2abc-b^2c+c^3$.

[25] $x^p+y^q+2z^r$. [26] $\frac{2}{3}a^3-\frac{4}{5}a^2b+\frac{1}{2}b^3$. [27] $\frac{3}{4}a+\frac{1}{4}b$.

[28] $x^4+x^3+x^2+x+1$. [29] $1+x+x^2+x^3+x^4+\ldots\ldots+x^{n-1}$.

[30] $(3ab+c)(a+b)$. [31] $(x^2+xy+y^2)(x^2-xy+y^2)$ and $(x+y)(x-y)(x^2+y^2)^2$ or $(x^3+x^2y+xy^2+y^3)(x^3-x^2y+xy^2-y^3)$.

[32] $\frac{a^4-2a^3+a^2+0{\cdot}15a+0{\cdot}02}{a(a-2)}=a^2+1+\frac{2{\cdot}15a+0{\cdot}02}{a(a-2)}$. [33] $12(x-1)$.

[34] $a-y$. [35] $\frac{1+x+x^2}{1-x-x^4+x^5}$. [36] $\frac{a+b+c}{a-b-c}$ and $\frac{a-2b}{a+b-c}$.

[37] $\frac{hkm(adf+bcf+bde)}{bdf(gkm+him+hkl)}$ and $\frac{a^2+2ab-b^2}{a^2+b^2}$. [38] $\frac{a+v}{a}$ and $\frac{a+2v}{a}$.

[39] $4a$. [40] $-\frac{a^2-(b+c+d)d+b(b+c)}{a(a^2-b^2)}$. [41] $\frac{(y-1)x^2+y(y+1)x-y^3}{x(x^3-y^3)}$.

[42] 1st product $=a-b$, the 2d $=x^2-xy+y^2$.

[43] $\frac{(a+b)x-c}{x}=a+b-\frac{c}{x}$ and $\frac{ab+x-y}{b}=a+\frac{x-y}{b}$. [44] a^3-b^3.

[5] $\frac{3(a-b)(z^3+z^2-z-1)}{3z^4-6z^2+3}$, $\frac{(a+b)(z^3+z^2-z-1)}{3z^4-6z^2+3}$, $\frac{3ab(z+1)^2}{3z^4-6z^2+3}$, $\frac{3a^2(z-1)^2}{3z^4-6z^2+3}$, and $\frac{b^2(z+1)^2}{3z^4-6z^2+3}$. [6] $\frac{(2a-b)^2}{4a^4-4a^3b+a^2b^2-2ab^3+b^4}$ and $\frac{(a-b)^2}{4a^4-4a^3b+a^2b^2-2ab^3+b^4}$.

EXERCISE XXXI.

[1] $\frac{x+y}{a}$. [2] $\frac{x-y+z-v-w}{a}$. [3] $\frac{a^2+2ab+b^2}{d}=\frac{(a+b)^2}{d}$. [4] $2x-3$.

[5] $\frac{(m-n)^2}{p-q}$. [6] $2a$. [7] $-ab\times\frac{(2a-b)^2}{a^2+2b^2}$. [8] $a\times\frac{x+y}{xy}$.

[9] $a\times\frac{y-z}{xy}$. [10] $\frac{ad+bc}{bd}$. [11] $\frac{bc+ac+ab}{abc}$.

[12] $\frac{30a^2qrs+20b^2prs-15c^2pqs+12d^2pqr}{60pqrs}$.

[13] $\frac{80xy^2p^2r^2+72x^2y^2zpr-100y^3z^3q-75xp^3qr^3}{120p^3qr^2}$. [14] $\frac{a^2+b^2}{a^2-b^2}=1+\frac{2b^2}{a^2-b^2}$.

[15] $\frac{(a+b)^2}{ab(a-b)}$. [16] $\frac{y}{x+y}$. [17] $\frac{5}{(4x-9)(3x-8)}$. [18] $-\frac{5x^2-8x+5}{x^2(x-1)}$.

[19] $\frac{5xy}{(3x-2y)(2x-3y)}$. [20] $\frac{bcd-acd+abd-abc}{abcd}=\frac{cd(b-a)+ab(d-c)}{abcd}$.

[21] $\frac{(a+b+c)x^2-2(ab+ac+bc)x+3abc}{x^3+(a+b+c)x^2+(ab+ac+bc)x-abc}$. [22] $\frac{6x^3+31x^2-66x+99}{x^2(x-3)^2}$.

[23] $\frac{791x^2-773x+192}{(7x-3)^3}$. [24] $\frac{16x^2+13xy+3y^2}{36(x+y)(x-y)^2}$. [25] $\frac{b^2}{a+b}$.

[26] $\frac{32x^3+37x^2-65x-60}{8x^4-6x^3-17x^2+6x+9}$. [27] $\frac{2b(10a^2+3b^2)}{5(a^2-b^2)}=\frac{2}{5}b\times\frac{10a^2+3b^2}{a^2-b^2}$.

[28] $2m+\frac{2n^4}{(m+n)(m^2+n^2)}=\frac{2(m^5-n^5)}{m^4-n^4}$. [29] $5(a-b)^2$.

[30] $12(a-b)-\frac{b(4a^2+13ab-9b^2)}{36a(a-b)}=\frac{432a^3-868a^2b+419ab^2+9b^3}{36a(a-b)}$.

EXERCISE XXXII.

[1] $\frac{ac}{bd}$. [2] $\frac{4ac}{5by}$. [3] $-\frac{15bx^2z}{4ay^2v}$. [4] $\frac{6p^3}{55qy}$. [5] $\frac{21a^2b^2}{x^2-8x+15}$.

[6] $\frac{x-a}{x+a}$. [7] $\frac{ab}{x^2-(a+b)x+ab}$. [8] $\frac{14ac^2(c-d)^2}{15b^2(a+b)(c+d)}$.

[9] $\frac{2a(a^3+b^3)}{3b(a-b)}$. [10] $\frac{a^2+2ab+b^2}{ab}$.

[11] $\frac{144a^4+31a^2b^2-14b^4}{12a^2}=12a^2+2\frac{7}{12}b^2-\frac{7b^4}{6a^2}$.

[12] $\frac{m^4n(p+q)}{pq(p-q)(p^2+q^2)}=\frac{m^4n(p+q)}{pq(p^3-p^2q+pq^2-q^3)}$. [13] a^3-b^3.

[14] $-\frac{(m+n)^2(3m^4+5n^4)}{m+2n}=-3m^5-3m^4n+9m^3n^2-18m^2n^3+41mn^4-72n^5+\frac{149n^6}{m+2n}$. [15] $\frac{9}{35}(x-y)$. [16] $\frac{9x^{n-2}y^{m-2}(a+b)}{a-b}$.

[17] $\frac{a^3+a^2b+ab^2+b^3}{a^3b^3}=\frac{1}{b^3}+\frac{1}{a^3}+\frac{1}{ab}\left\{\frac{1}{a}+\frac{1}{b}\right\}$.

[18] $\frac{(x-1)(y-1)(z-1)}{xyz}=\frac{xyz-(xy+xz+yz)+(x+y+z)-1}{xyz}$.

EXERCISE XXXIII.

[1] $\frac{ay}{bx}$. [2] $\frac{3bx^2}{5xy^2}$. [3] $\frac{45a^4}{b}$. [4] $\frac{2x}{45x^3}$. [5] $\frac{3(x+y)}{2x^2y^2}$. [6] $\frac{a(a+y)}{a-b}$.

[7] $\frac{a^2-ab+b^2}{3(a-b)}$. [8] $\frac{(3x+y)(x+y)}{(x+3y)(x-y)}=\frac{3x^2+4xy+y^2}{x^2+2xy-3y^2}$. [9] $\frac{x}{y}$.

[10] $3\times\frac{5a+7y}{5a-7y}$. [11] $\frac{19}{(19x-y)(x-y)}$. [12] $\frac{15(x^2+xy+y^2)}{7xy}$.

[13] $\frac{a^2p}{b(aq+bq)}$. [14] $\frac{5x^{p-3}y^{q-2}}{4(a+b)}$. [15] $\frac{p^2+pq+3q^2}{p^2-2pq-3q^2}$.

[16] 1. [17] $\frac{apq}{2c}$. [18] $\frac{ab+c-d}{b}$. [19] $\frac{4xy}{(x^2-y^2)^2}$. [20] 1.

EXERCISE XXXIV.

[1] 1200. [2] $\frac{1075}{1296}$. [3] 2304. [4] $\frac{133}{384}$. [5] $-920\frac{23}{39}$.

[6] $12x^2+18xy-11y^2$. [7] $22a^4+4a^3b-16a^2b^2-4ab^3+2b^4+12d^3\sqrt{xy}+(3d^3-2e^3)\sqrt{ab}$. [8] $10x^3-16x^2y+3xy^2-3y^3+7p^2q-6h^2\sqrt{xy}-4m^3$. [9] $x^2+3xy-10y^2+(3a-2b)x-(a-b)y-ab(5c-a)$.

[10] $2a^4-a^3b-2a^2b^2-9ab^3+7b^4+3x^2y^2$. [11] $\frac{1}{12}(5a-36b-37x)$.

[12] $-(7x^3+9x^2y-14xy^2-6y^3)$. [13] $\frac{3}{4}x^6y^2z^2(a+x)^{10}(a+y)^5$.

[14] $-(\frac{7}{10}a^7y^2-\frac{2}{5}a^6y^3+\frac{14}{9}a^5y^4-\frac{28}{75}a^4y^5+\frac{14}{45}a^3y^6+\frac{7}{9}a^2y^7)$.

[15] $21x^7-14x^6y+22x^5y^2+9x^4y^3+10x^3y^4-24x^2y^5+10xy^6+8y^7$.

[16] $x^4+x^2y^2+y^4$. [17] $x^6+x^4y^2-x^2y^4-y^6$.

[18] $5x^4+3\frac{1}{2}ax^3-8\frac{1}{12}a^2x^2+\frac{5}{6}a^3x+1\frac{1}{6}a^4$. [19] $4a^3-5a^2b-3ac^2+6c^3$.

[20] $A=(a^2+b^2)(x^2+y^2)$. $C=(a^2+b^2+c^2)(x^2+y^2+z^2)$.
$B=(a^2+b^2+c^2)(x^2+y^2)$. $D=(a^2+b^2+c^2+d^2)(x^2+y^2+z^2+v^2)$.

[21] $\frac{1}{12}x(249x+242)$. [22] $10a^2by-10b^3y-16ax^3y+20ax^2y^4-20ay^6$.

[23] $a^{2p}-b^{2q}-c^{2r}+2b^qc^r$. [24] $ab^2+2abc-b^2c+c^3$.

[25] $x^p+y^q+2z^r$. [26] $\frac{2}{3}a^3-\frac{4}{5}a^2b+\frac{1}{2}b^3$. [27] $\frac{3}{7}a+\frac{1}{4}b$.

[28] $x^4+x^3+x^2+x+1$. [29] $1+x+x^2+x^3+x^4+\ldots\ldots+x^{m-1}$.

[30] $(3ab+c)(a+b)$. [31] $(x^2+xy+y^2)(x^2-xy+y^2)$ and $(x+y)(x-y)(x^2+y^2)^2$ or $(x^3+x^2y+xy^2+y^3)(x^3-x^2y+xy^2-y^3)$.

[32] $\frac{a^4-2a^3+a^2+0{\cdot}15a+0{\cdot}02}{a(a-2)}=a^2+1+\frac{2{\cdot}15a+0{\cdot}02}{a(a-2)}$. [33] $12(x-1)$.

[34] $a-y$. [35] $\frac{1+x+x^2}{1-x-x^4+x^5}$. [36] $\frac{a+b+c}{a-b-c}$ and $\frac{a-2b}{a+b-c}$.

[37] $\frac{hkm(adf+bcf+bde)}{bdf(gkm+him+hkl)}$ and $\frac{a^2+2ab-b^2}{a^2+b^2}$. [38] $\frac{a+v}{a}$ and $\frac{a+2v}{a}$.

[39] $4a$. [40] $-\frac{a^2-(b+c+d)d+b(b+c)}{a(a^2-b^2)}$. [41] $\frac{(y-1)x^2+y(y+1)x-y^3}{x(x^3-y^3)}$.

[42] 1st product $=a-b$, the 2d $=x^2-xy+y^2$.

[43] $\frac{(a+b)x-c}{x}=a+b-\frac{c}{x}$ and $\frac{ab+x-y}{b}=a+\frac{x-y}{b}$. [44] a^3-b^3.

[45] $y=\frac{2c^3-a^3}{b(3a+2c)}$. [46] $\frac{p^4+p^3+p^2+p+1}{(p-1)^4}=\frac{(p^5-1)+(p-1)}{(p-1)^4}=\frac{(p^5-1)}{(p-1)^5}$.
[47] $\frac{x^4-y^4}{x-y}=\frac{(x-y)(x^3+x^2y+xy^2+y^3)}{x-y}=\frac{4x^3}{1}=4x^3$, $\frac{x^2-y^2}{(x-y)^2}=\frac{(x-y)(x+y)}{(x-y)(x-y)}=$
$\frac{2x}{0}=\infty$, $\frac{x^2+xy+y^2}{x^3-y^3}=\frac{1\times(x^2+xy+y^2)}{(x-y)(x^2+xy+y^2)}=\frac{1}{0}=\infty$. [48] $3a^2+2ab-b^2$.
[49] 1. [50] $(a+b)^2-(c^2-d^2)$.

EXERCISE XXXV.

[1] $x=a-2b+3c$. [2] $x=1{\cdot}23456$. [3] $x=-9{\cdot}876$.
[4] $x=3a-3b+3c$. [5] $x=p-2q+3r-4s$. [6] $x=a-b$.
[7] $x=7$. [8] $x=e:d$. [9] $x=0{\cdot}03349076$.
[10] $x=84\frac{1}{2}$. [11] $x=1\frac{3}{5}$. [12] $x=a-b$.
[13] $x=a^2+ab+b^2$. [14] $x=\frac{p\mp n}{m}$. [15] $x=\frac{n}{q\mp p}$.
[16] $x=7{\cdot}77$. [17] $x=m+n$. [18] $x=10\frac{11}{12}$.
[19] $x=140$. [20] $x=\frac{-n+1-m+p}{m-p-n+1}$. [21] $x=1\frac{407}{2275}$.
[22] $x=-1\frac{241}{326}$. [23] $x=\frac{pa}{q\mp p}$. [24] $\frac{aehn+dbhn-gben-mheb}{behn(c+f-k-o)}$.
[25] $x=\frac{11}{432}$. [26] $x=\frac{(n+m)(p+q)}{1-p-q}$. [27] $x=\frac{2n}{a-b}$.
[28] $x=0{\cdot}5$. [29] $x=1$. [30] $x=1{\cdot}1$.
[31] $x=\frac{2a(3b-4c)}{5b-6c}$. [32] $x=4\frac{1}{3}$. [33] $x=a^2b^2-a-b$.
[34] $x=6$. [35] $x=a-b$. [36] $x=26\frac{115}{143}$.
[37] $x=\frac{2}{3}$. [38] $x=7$. [39] $x=11$.
[40] $x=\frac{ab(a+b)}{a^3+a^2b-3ab^2-b^3}$. [41] $x=\frac{1}{10}$. [42] $x=\frac{1}{4a(a+b)}$.
[43] $x=\frac{83}{84}$. [44] $x=3$. [45] $x=\frac{2}{3}$.
[46] $x=\frac{ab}{a+b}$. [47] $x=5$. [48] $x=1$.

EXERCISE XXXVI.

[1] 25 years. [2] $6\frac{16}{19}$ per cent. [3] 765 acres of beeches, 68 of oaks, and 461 of pines. [4] 5800*l.*

[5] 12000*l.* [6] 12 years. [7] The first $13\frac{3}{4}$ per cent, the second $12\frac{1}{2}$ per cent. [8] 2110*l.*

[9] 142857. [10] 857142. [11] 290.

[12] $\frac{1}{2}$. [13] 200 oxen. [14] 28 persons.

[15] $\frac{1}{4}$ mile. [16] In 2 years 8 times as old, in 5 years 5 times, and $1\frac{1}{2}$ years ago 57 times. [17] 30 years.

[18] D is 7 miles from E. [19] 90 hanks.

[20] 8 gallons per minute must flow into it, and it contains 240 gallons.

[21] 816 cubic feet. [22] Loses 4 per cent.

[23] Either the gain is $\frac{p'(100+n)-100p}{p}$ per cent, or the loss is $\frac{100p-p'(100+n)}{p}$ per cent, according as $p'(100+n) \gtrless 100p$. There is neither gain nor loss when $p'(100+n)=100p$.

[24] After 6 days, at a distance of $31\frac{1}{2}$ miles. [25] 4 leagues.

[26] $3\frac{3}{4}$ hours. [27] Either after 1 hour 12 min., or after 13 hours 12 min.; in the 1st case before their meeting, in the 2d after it.

[28] 11 times. 1st, at $32\frac{8}{11}$ min. past 12; 2d, $38\frac{2}{11}$ past 1; 3d, $43\frac{7}{11}$ past 2; 4th, $49\frac{1}{11}$ past 3; 5th, $54\frac{6}{11}$ past 4; 6th, 6 o'clock, and so on, each time 1 hour $5\frac{5}{11}$ min. later than the preceding. [29] 600 leaps.

[30] 2 hours 24 min. [31] $26\frac{1}{4}$ days. [32] The 1st 16 rods, the 2d 18 rods. [33] 8 horses. [34] The 1st 30 H.-P., the 2d 24.

[35] 10 months. [36] 428*l.* [37] 3 months. [38] 1000*l.*

[39] A, 40*l.*; B, 48*l.*; C, 48*l.* [40] A, 3*l.*; B, 1*l.* 16*s.*; C, 2*l.*

[41] A, 1*l.* 16*s.*; B, 2*l.* 6*s.* 6*d.* [42] 9. [43] 19.

[44] 3 must be deducted from both. [45] To both we must either add $\frac{aq-bp}{p-q}$ or $\frac{bp-aq}{q-p}$, or subtract $\frac{bp-aq}{p-q}$ or $\frac{aq-bp}{q-p}$. [46] 55 miles.

[47] C had lost 5*l.* [48] 75 lbs. [49] One is 28, the other 42.

[50] 200 shillings.

EXERCISE XXXVII.

[1] $x=\frac{s+d}{2}$, $y=\frac{s-d}{2}$. [2] $x=7\frac{8}{9}$, $y=10\frac{11}{12}$. [3] $x=0$, $y=p$.

[4] $x=11$, $y=17$. [5] $x=1\cdot543689$, $y=0\cdot8392867$.

[6] $x=37$, $y=43$. [7] $x=\frac{a}{a-b}$, $y=\frac{b}{a+b}$.

[8] $x=\frac{2}{m+n}$, $y=\frac{2}{m-n}$. [9] $x=na-a$, $y=a+b$.

[10] $x=7\frac{3}{4}$, $y=9\frac{2}{3}$. [11] $x=1\cdot2$, $y=3\cdot4$.

[12] $x=-5\frac{1}{2}$, $y=-9\frac{1}{2}$. [13] $x=a+b$, $y=a-b$.

[14] $x=a-2b+3c$, $y=3a-2b+c$. [15] $x=-9\frac{678}{925}$, $y=-20\frac{662}{925}$.

[16] $x=14$, $y=46$. [17] $x=1$, $y=3$.

[18] $x=\frac{(q^2-p^2)n-(m^2-n^2)p}{nq-mp}$, $y=\frac{(m^2-n^2)q-(q^2-p^2)m}{nq-mp}$.

[19] $x=a-2b+3c$, $y=a+2b-3c$. [20] $x=1\frac{1}{2}$, $y=2\frac{1}{2}$.

[21] $x=29$, $y=28$. [22] $x=2$, $y=2$. [23] $x=17$, $y=22$, $z=45$.

[24] $x=64$, $y=80$, $z=100$. [25] $x=48$, $y=54$, $z=64$.

[26] $x=13$, $y=24$, $z=62$. [27] $x=\frac{2}{a+b-c}$, $y=\frac{2}{a-b+c}$, $z=\frac{2}{b+c-a}$.

[28] $x=6$, $y=9$, $z=\frac{1}{3}$. [29] $x=16$, $y=4$, $z=4$.

[30] $x=\frac{lmn(bde+acf)}{cfmn-bfln+bdlm}$, $y=\frac{lmn(bde+acf)}{afln+demn-adlm}$, $z=\frac{lmn(bde+acf)}{beln-cemn+aclm}$.

EXERCISE XXXVIII.

[1] 33 for, and 15 against it. [2] One had 4, the other 8.
[3] The first 18*d.*, the second 14*d.* [4] $\frac{5}{24}$. [5] $\frac{3}{7}$.
[6] A, 7000*l.*; B, 3000*l.* [7] 40 and 7.
[8] $\frac{a^2}{a+1}$ and $\frac{a}{a+1}$. [9] 1234 and 5678.
[10] a^2+b and $a+b^2$. [11] The father is 42; the son, 12.
[12] The capital is 4700*l.*; interest, $4\frac{3}{4}$ per cent.
[13] A had 40*l.*; B, 80*l.* [14] In the one, 100 quarts; in the other, 50.
[15] In the one, $\frac{11}{16}n$ quarts; in the other, $\frac{21}{16}n$ quarts.
[16] The first, 3; the other, 7. [17] 16 and 4. [18] $\frac{(a+b)^2}{2(a-b)}$ and $\frac{a+b}{2}$.
[19] The first 480, the second 400, the third 560 gallons.
[20] The one $\frac{d}{2}\left(\frac{1}{m}+\frac{1}{n}\right)$, the other $\frac{d}{2}\left(\frac{1}{m}-\frac{1}{n}\right)$ feet per second.
[21] B, $\frac{d(u+n-m)}{t(u+n)+mu}$ feet; C, $\frac{d(t+m-n)}{t(u+n)+mu}$ feet per second.
[22] A, 3100*l.*; B, 7400*l.* [23] In the proportion 7 : 8 : 9.
[24] He has 300 oxen, and fodder for 60 days.
[25] 9 yards of 8 quarters wide. [26] $\frac{2b}{1-a}$ and $\frac{2b}{1+a}$.
[27] 9 and 6. [28] A had 1*s.* 8*d.*; B, 3*s.*; C, 5*s.* 8*d.*; D, 11*s.*
[29] The 1st, 449; 2d, 225; 3d, 113; 4th, 57; 5th, 29; 6th, 15; 7th, 8.
[30] The 1st is $\frac{2^{n-4}(n+1)}{2^n}a$; the 2d, $\frac{2^{n-2}(n+1)}{2n}a$; 3d, $\frac{2^{n-3}(n+1)}{2n}a$, &c.

INDICES.

EXERCISE XXXIX.

[1] $a^{\frac{1}{2}}, a^{\frac{1}{3}}, a^{\frac{2}{3}}, a^{\frac{1}{5}}$. [2] $xy^{\frac{1}{3}}z, a^{\frac{2}{3}}bc^{\frac{1}{2}}, x^{\frac{1}{2}}y^{\frac{1}{3}}z, a^{\frac{3}{2}}b^2$.
[3] $xy^{\frac{3}{2}}, a^{\frac{1}{4}}, a^{\frac{1}{2}}b^{\frac{3}{2}}c^{\frac{1}{4}}, ab^{\frac{1}{2}}c^{\frac{1}{4}}$. [4] $x^{\frac{1}{2}}y^{\frac{1}{4}}, xy^{\frac{3}{4}}, x^{\frac{2}{5}}y^2, x^2y^{\frac{1}{3}}$.
[5] (i) $\frac{1}{x}, \frac{1}{c^2}, \frac{1}{b^4}, \frac{2}{a^{-1}c^2}, \frac{4}{x^{-2}y^2}, \frac{5}{a^{-3}c^3}, \frac{2}{3x^{-1}y^2}$.
(ii) $x^{-1}, c^{-2}, b^{-4}, 2ac^{-2}, 4x^2y^{-2}, 5a^3c^{-3}, \frac{2}{3}xy^{-2}$.
[6] (i) $\frac{1}{x^{-3}y^2z}, \frac{3}{2x^{-2}yz}, \frac{3}{a^{-1}b^{-1}c^{-1}d}, \frac{1}{2a^{\frac{1}{2}}b^{-1}c^{-2}}, \frac{1}{3a^{\frac{2}{3}}b^{\frac{1}{3}}c^{-1}}$.
(ii) $x^3y^{-2}z^{-1}, \frac{3}{2}x^2y^{-1}z^{-1}, 3abcd^{-1}, \frac{1}{2}a^{-\frac{1}{2}}bc^2, \frac{1}{3}a^{-\frac{2}{3}}b^{-\frac{1}{3}}c$.

EXERCISE XL.

[1] $\sqrt[3]{x}, \sqrt[3]{3x^2}, \sqrt[5]{4a^4}, \sqrt[6]{7b}, \sqrt[4]{y^3}$.
[2] $\frac{\sqrt{x}}{\sqrt[3]{y}}, \frac{\sqrt{x}\sqrt[3]{y}}{3\sqrt{ab}}, \frac{4\sqrt[3]{a}\sqrt{b}}{6\sqrt[5]{c^4}}, \frac{\sqrt[5]{a^4b}}{\sqrt{a}\sqrt[5]{b^2}}, \frac{\sqrt[3]{x}\sqrt[5]{y^3}}{4\sqrt[3]{a}\sqrt{b}}$.

EXERCISE XLI.

[1] $\frac{x}{y}$, $\frac{b}{a^2c}$, $\frac{1}{xyz^3}$, $\frac{x^2}{yz^3}$. [2] $\frac{1}{\sqrt[4]{x^3}}$, $\frac{\sqrt[3]{x}}{\sqrt{y}}$, $\frac{\sqrt[5]{y^4}}{\sqrt[3]{x^2}}$, $\frac{\sqrt[4]{x^3}}{\sqrt{y}}$.

[3] $\frac{ay}{xz^2}$, $3xy^2z$, $\frac{4a^3b}{c}$, xy^2z^3. [4] $\frac{\sqrt[3]{b}}{a^3}$, $\frac{\sqrt{y}}{\sqrt[4]{x^3}}$, $\frac{y}{\sqrt[5]{x^3}}$, $\frac{b^3}{\sqrt[5]{a}}$.

EXERCISE XLII.

[1] $\sqrt[nq]{a^{mp}}$, $\frac{1}{\sqrt[nq]{a^{mp}}}$, $\frac{1}{\sqrt[nq]{a^{mp}}}$, $\sqrt[nq]{a^{mp}}$. [2] $\sqrt[36]{a^9b^8}$, $\sqrt[40]{\frac{b^5}{a^{20}c^4}}$.

[3] $\sqrt[20]{\frac{1}{a}}$, $a^3b\sqrt[6]{c}$. [4] $\sqrt[6]{\frac{(a+b)^3}{c^4d^2}}$, $\sqrt[8]{a^4b}$. [5] $\frac{1}{cd^m}\sqrt[24]{\frac{(c-d)^3(a+x^3)^8}{c^{12}d^{6m}}}$.

EXERCISE XLIII.

[1] $\sqrt[60]{7}$. [2] $bc(\frac{5}{7}ac)^{\frac{1}{2}}$. [3] $16\sqrt{2}$. [4] $4a^2+16ac^{\frac{1}{2}}-9b+16c^{\frac{1}{2}}$.

[5] $a^{\frac{3}{2}}-b^{\frac{4}{5}}=a\sqrt{a}-\sqrt[5]{b^4}$, $(5a^2-41ab+42b^2)\sqrt[12]{a}$.

[6] $\left(1-\frac{2bc}{a}-\frac{b}{a}+\frac{2b^2c}{a^2}\right)\sqrt[30]{\frac{1}{a^3b^{10}}}$. [7] $\frac{b^2}{c^2}\sqrt[5]{\frac{a^2d^2}{f^2}}-c^2d^2\sqrt[3]{\frac{a^2c^2}{b^2g^2}}$.

[8] $a^{\frac{1}{2}}-2a^{\frac{1}{4}}b^{\frac{1}{4}}$. [9] $\frac{a^2}{c}\sqrt[4]{b^3}$, $\frac{c}{d\sqrt[12]{a}}$. [10] $a^{\frac{1}{3}}b^{\frac{1}{4}}-ca^{\frac{1}{5}}b^{\frac{1}{3}}=\sqrt[12]{a^4b^3}-c\sqrt[15]{a^3b^5}$.

[11] $5a^{\frac{7}{4}}-6a^{\frac{3}{4}}b=(5a-6b)\sqrt[4]{a^3}$. [12] $a^{\frac{1}{2}}+a^{\frac{1}{4}}b^{\frac{1}{4}}+b^{\frac{1}{2}}=\sqrt{a}+\sqrt{b}+\sqrt[4]{ab}$.

SURDS.

EXERCISE XLIV.

[1] $9^{\frac{1}{2}}$, $16^{\frac{1}{2}}$, $(\frac{1}{25})^{\frac{1}{2}}$, $(\frac{27}{8})^{\frac{1}{3}}$. [2] $25^{\frac{1}{2}}$, $125^{\frac{1}{3}}$, $36^{\frac{1}{2}}$, $216^{\frac{1}{3}}$, $49^{\frac{1}{2}}$, $343^{\frac{1}{3}}$, $64^{\frac{1}{2}}$, $512^{\frac{1}{3}}$.

[3] $\left(\frac{x^3+3x^2y+3xy^2+y^3}{27}\right)^{\frac{1}{3}}$. [4] $(625x^8y^4)^{\frac{1}{4}}$, $\left(\frac{a^4+4a^3b+6a^2b^2+4ab^3+b^4}{16}\right)^{\frac{1}{4}}$.

[5] $\sqrt{12}$, $\sqrt{6}$, $\sqrt{\frac{9}{2}}$, $\sqrt{\frac{8}{5}}$. [6] $\sqrt{\frac{5}{6}}$, $\sqrt[3]{256}$, $\sqrt[3]{72}$, $\sqrt[4]{78125}$.

[7] $\sqrt[4]{\frac{1}{4}}$, $\sqrt[3]{\frac{128}{243}}$, $\sqrt{16x}$, $\sqrt{27x^2y}$, $\frac{1}{2y}$, $\frac{1}{\sqrt{(a^2+ab+b^2)}}$.

[8] $3\sqrt{6}$, $5\sqrt{3}$, $4\sqrt{6}$, $8\sqrt{2}$. [9] $36\sqrt{3}$, $4\sqrt[3]{2}$, $7\sqrt[3]{2}$, $12\sqrt[3]{2}$.

[10] $8\sqrt[3]{7}$, $\frac{1}{2}\sqrt{2}$, $\sqrt{6}$, $\sqrt[3]{50}$. [11] $3\sqrt[3]{3}$, $24\sqrt{3}$, $\frac{1}{3}\sqrt[3]{4}$, $\frac{2}{5}\sqrt{2}$.

EXERCISE XLV.

[1] $(b+c+d)\sqrt[m]{a}$, $\sqrt[6]{5}$. [2] $12\sqrt{7}-3\sqrt{6}+\frac{3}{2}\sqrt[4]{11}-5\sqrt[3]{13}$.

[3] $4\sqrt{6}$, $8\sqrt{2}$. [4] $8\sqrt[3]{2}$. [5] $\frac{28}{15}\sqrt{15}$. [6] $-13\sqrt{3}$.

[7] $\frac{29\sqrt{3}}{2}$. [8] $8\sqrt{7}-\sqrt[3]{3}$. [9] $\frac{27}{4}\sqrt[3]{2}$. [10] $(a-c)\sqrt{5c}$.

EXERCISE XLVI.

[1] $abc\sqrt[n]{xyz}$. [2] 140. [3] $\sqrt[mn]{a^nb^m}$.

[4] $\frac{7}{2}\sqrt[3]{120}$, $10+4\sqrt{35}+6\sqrt{50}$. [5] $39\sqrt{2}-16\sqrt{5}$, $1+\sqrt{6}$.

[6] $\sqrt[m]{\frac{a}{b}}$, $\frac{c}{d}\sqrt[m]{\frac{a}{b}}$, $\sqrt[m]{\frac{a^m}{b}}$. [7] $\sqrt{a}$, $\frac{1}{2}\sqrt[3]{\frac{b^5c^4}{d}}$. [8] $2\sqrt[6]{\frac{16}{3}}$, $5-\sqrt{2}$.

[9] $\sqrt{2}+4\sqrt{6}-\sqrt{3}-8\sqrt{\frac{2}{3}}$. [10] $\sqrt{6}+\sqrt{2}-\sqrt[6]{\frac{3}{2}}$.

EXERCISE XLVII.

[1] $2-\sqrt{3}$, $3\sqrt{2}-3$, $15+3\sqrt{21}$. [2] $\frac{1}{2}(\sqrt{2}+1)$, $\sqrt{15}+\frac{3}{2}\sqrt{6}$.

[3] $\frac{1}{10}$, $9+\frac{5}{2}\sqrt{10}$. [4] $\frac{a+b+2\sqrt{ab}+2\sqrt[4]{a^3b}+2\sqrt[4]{ab^3}}{a-b}$.

EXERCISE XLVIII.

[1] $2+\sqrt{3}$. [2] $5-3\sqrt{2}$. [3] $\sqrt{3}-\sqrt{2}$. [4] $\sqrt{2}-1$.

[5] $3\sqrt{7}-2\sqrt{6}$. [6] $1+\frac{1}{2}\sqrt{2}$. [7] $\frac{1}{2}\sqrt{6}+\frac{1}{2}\sqrt{2}$. [8] $5+\sqrt{3}$.

[9] $\sqrt[4]{18}-\sqrt[4]{2}$. [10] $\sqrt[4]{20}+\sqrt[4]{5}$. [11] $\sqrt[4]{8}-\sqrt[4]{2}$. [12] $\sqrt{(x-1)}-1$.

[13] $\sqrt{(a+b)}+\sqrt{(a-b)}$. [14] $\frac{c+(a^2-c^2)}{2}$. [15] $(y-1)\sqrt{x}$.

[16] $\sqrt{(ap-a^2)}-a$. [17] $\frac{ac}{d}+\sqrt{\left(\frac{3a}{b}-\frac{a^2c^2}{d^2}\right)}$.

[18] $\sqrt{ab}+\sqrt{\left(b^2-2ab+\frac{a^2}{4}\right)}$.

Miscellaneous Exercises.

[1] $\frac{1}{5}\sqrt{7}$, $\frac{2}{15}\sqrt{21}$, $\frac{1}{3}\sqrt[3]{5}$, $\frac{29}{2}\sqrt{3}$ or $\frac{29\sqrt{3}}{2}$.

[2] $\frac{2ax}{3b}\sqrt[4]{\left(\frac{ax^3+a^2x^2}{c^2-2b}\right)}$, $\frac{2a}{b}\sqrt[5]{\left(\frac{a^4+4ab^3-5b^5}{3b}\right)}$.

[3] $\left(3ab^2+2a^2-\frac{c^4}{b}\right)\sqrt{ac}$, $\left(\frac{5a}{b}+\frac{bc}{a}\right)\sqrt[3]{a^2b^2}$, $(3a^2b-\frac{2b^2}{a}+a^{m+3}+c)\sqrt[3]{2a^m}$.

[4] $a\sqrt{(c+d)}$, $a^{2m}\sqrt[3]{(b-a^mf^2)}$. [5] $\frac{\sqrt{ac}}{a+b}$, $\frac{x+1}{a}\sqrt{\frac{x}{a+b}}$.

[6] $(x+y+z)\sqrt{(a+b+c)}$, $(a+x+y+3)\sqrt{x}$, $\frac{a}{2d}\sqrt{ax}$.

[7] $(3a+8)\sqrt[4]{(b^2-c^2)}$, $(b+c-3a^3)\sqrt[5]{ab^2}-5ab\sqrt[4]{ab^2}$, $5\frac{5}{6}\sqrt{3}$, $\sqrt[3]{6}$.

[8] $12+3\sqrt[3]{20}-6\sqrt[3]{24}-\sqrt[3]{180}$, $(2p+q)\sqrt[12]{\frac{(a-c)^7}{a+c}}$.

[9] $5\sqrt{6}+5\sqrt{5}+2\sqrt[4]{125}+2\sqrt[4]{180}+2\sqrt[6]{54}+\sqrt[6]{2000}$, $a-c^2\sqrt[3]{b^2}$, $2a+12c\sqrt[3]{b^2}+(3c+8)\sqrt[6]{a^3b^2}$.

[10] $\frac{2}{q}\sqrt{2p}$, $\frac{1}{c}\sqrt[3]{\frac{p^2-pq+q^2}{p-q}}$, $\frac{4b}{c}\sqrt[4]{\frac{10a^3bd^2}{5c-d}}$, $\frac{a-b}{d}\sqrt[5]{\frac{2d}{3c}}$, $b+c\sqrt[m+1]{\frac{1}{b^{m+3}}}$.

[11] $a+b-x\sqrt{y}$, $\sqrt{xy}+\sqrt{(x^2-y^2)}$, $\frac{3}{2}\sqrt{3}-1+\frac{1}{2}\sqrt{2}-\frac{1}{2}\sqrt{\frac{49}{5}}$.

[12] $4\sqrt{7}-7$, $2\sqrt{5}-2\sqrt{3}$, $1\frac{2}{3}\sqrt{11}+1\frac{2}{3}\sqrt{2}$, $2\sqrt{21}-6$.

[13] $\sqrt{6}+\sqrt{3}$, $\frac{1}{2}\sqrt{6}+\frac{2}{3}\sqrt{3}$, $3\sqrt{7}+\sqrt{13}$, $\frac{3\{\sqrt{3}-\sqrt{2}+\frac{1}{3}\sqrt{(6-1)}\}}{2}$.

[14] $\sqrt[6]{x^5yz^3}$, $\sqrt[6]{ab^8c^2d^5}$, $2\sqrt[6]{8a}$, $\sqrt[30]{671088864b}$, $\sqrt[3]{(2a+5b)}$, $\sqrt[5]{(a+2b)}$.

[15] $\sqrt{40}+\sqrt{27}$, $\sqrt{(a+b)}-\sqrt{(a-b)}$, $\sqrt[4]{24}+\sqrt[4]{6}$, $(\sqrt{y}-1)\sqrt{x}$, $\pm(2+\sqrt{3})$, $3\sqrt{6}$.

QUADRATICS.

EXERCISE XLIX.

[1] $x=\pm 3$. [2] $x=\pm 5$. [3] $x=\pm\frac{c+b}{a}$. [4] $x=\pm 8$.
[5] $x=\pm 2$. [6] $x=\pm 1$. [7] $x=\pm\frac{1}{2}$. [8] $x=\pm 6$.
[9] $x=\pm 9$. [10] $x=\pm\sqrt{\frac{2c+d}{a-b}}$.

EXERCISE L.

[1] $x=6$ and 4. [2] $x=7$ and -1. [3] $x=5$ and -7.
[4] $x=10$ and -2. [5] $x=370$ and -430. [6] $x=7$ and 1.
[7] $x=6\frac{1}{2}$ and $\frac{1}{2}$. [8] $x=8$ and $-2\frac{1}{4}$. [9] $x=5$ and $-4\frac{1}{3}$.
[10] $x=60\frac{2}{3}$ and $16\frac{2}{7}$. [11] $x=5\frac{17}{20}$ and $3\frac{3}{4}$. [12] $x=-2\frac{1}{7}$ and $5\frac{1}{2}$.

EXERCISE LI.

[1] $x=-25\frac{10}{39}$ and -52. [2] $x=4+\sqrt{30}$ and $4-\sqrt{30}$.
[3] $x=\frac{-1+\sqrt{85}}{6}$ and $\frac{-1-\sqrt{85}}{6}$. [4] $x=\frac{118+\sqrt{13724}}{5}$ and $\frac{118-\sqrt{13724}}{5}$.
[5] $x=-46$ and $25\frac{1}{4}$. [6] $x=14$ and -10. [7] $x=10$ and $-\frac{2}{3}$.
[8] $x=67\frac{1}{6}$ and $4\frac{1}{2}$. [9] $x=2$ and -3. [10] $x=7\frac{23}{113}$ and $2\frac{1}{2}$.
[11] $x=\frac{d}{c}$ and $-\frac{b}{a}$. [12] $x=11$ and 7. [13] $x=4$ and $\frac{8}{3}$.
[14] $x=2a+b$ and 0. [15] $x=27a$ and $3a$. [16] $x=-1\pm\frac{3}{2\sqrt{2}}$.

EXERCISE LII.

[1] $x=3\frac{1}{3}$ and -4. [2] $x=2\frac{1}{3}$ and $3\frac{1}{2}$. [3] $x=1\pm 3\sqrt{-1}$.
[4] $x=13\frac{22}{31}$ and 8. [5] $x=\frac{1}{16}(7\pm\sqrt{-1039})$. [6] $3\frac{1}{2}$ and $2\frac{2}{3}$.

EXERCISE LIII.

[1] $(x-7)(x+9)$. [2] $(x-12)(2x+19)$.
[3] $(x+1)(5x-8)$. [4] $(2x-\sqrt{2})(3x+2\sqrt{3})$.
[5] $(3x-2\sqrt{5})(x+\sqrt{5})$. [6] $(ax+b)(bx-a)$.
[7] $(x+1)(2x+3a-2)$. [8] $(x+1)\{(m-2)x-(m-5)\}$.
[9] $(5m+3)(3m-5)$. [10] $(3m+7)(3m-7)$.
[11] $(a-b\sqrt{-1})(a+b\sqrt{-1})$. [12] $(abx+c^2)(cx-ab)$.

EXERCISE LIV.

[1] $x^2+x=30.$ [2] $9x^2+9x=4.$ [3] $x^2-9=0.$

[4] $x^2-\frac{24}{5}x=1.$ [5] $x^2-\frac{5}{2}x=6.$ [6] $x^2-4x=-1.$

[7] $x^2-6\frac{1}{6}x+9\frac{1}{2}=0.$ [8] $11x^2+x-180=0.$ [9] $x^3-49x-120=0.$

[10] $x^3-18x^2+87x-110=0.$ [11] $x^4-45x^2-40x+84=0.$

[12] $x^3+\frac{8}{5}x^2+\frac{9}{100}x-\frac{9}{100}=0.$ [13] $x^4-\frac{19}{4}x^3+\frac{49}{8}x^2-\frac{11}{4}x+\frac{3}{8}=0.$

[14] $x^3-13x^2+49x-45=0.$ [15] $x^5-3x^4-8x^3+24x^2-9x+27=0.$

EXERCISE LV.

[1] $x=\pm 7$ or $\pm 5.$ [2] $x=\pm\sqrt{2}$ or $\pm\sqrt{-1}.$

[3] $x=\pm 4$ or $\pm 3.$ [4] $x=\pm 3$ or $\frac{1}{21}\sqrt{-7035}.$

[5] $x=\pm\{\sqrt{(bc+a^2)}+a\}$ or $\pm\{\sqrt{(bc+a^2)}-a\}.$

[6] $x=\pm a(\sqrt{6}+\sqrt{2})$ or $\pm a(\sqrt{6}-\sqrt{2}).$ [7] $x=3$ or $-\sqrt[3]{41}.$

[8] $x=\pm\sqrt[3]{(-3+\sqrt{10})}.$ [9] $x=\sqrt[3]{(54\pm 30\sqrt{3})}.$

[10] $3+\sqrt{2}$ or $3-\sqrt{2}.$

[11] $x=\sqrt{(1\pm\sqrt{2})}$ or $\sqrt[4]{(1+\sqrt{2})}$ and $-\sqrt[4]{(1+\sqrt{2})}.$

[12] $x=3$ or $\sqrt[5]{214}.$ [13] $x=\frac{1}{2}\sqrt{\{-2b^2\pm 2\sqrt{(4a^4+b^4)}\}}.$

[14] $x=36$ or $35.$ [15] $x=25$ or $20\frac{1}{4}.$

[16] $x=16$ or $18\frac{7}{9}.$ [17] $x=\pm 2$ or $\pm\sqrt{-1}.$

[18] $x=\pm\sqrt{\{a^2-b+\sqrt{(b^2+c^2)}\}}$ or $\pm\sqrt{\{a^2-b-\sqrt{(b^2+c^2)}\}}.$

[19] $x=-2\frac{14}{65}.$ [20] $x=\frac{1\pm\sqrt{-3}}{2}={\cdot}5\pm{\cdot}8664\sqrt{-1}.$

[21] $x=\sqrt{-1}\pm\sqrt{-2}.$ [22] $x=(m-n)^2$ and $(m+n)^2.$

[23] $x=8\frac{1}{3}$ and $-4\frac{1}{7}.$ [24] $x=\sqrt{-7}$ and $\sqrt{-11}.$

[25] $x=13\frac{14}{15}$ and $-1\frac{101}{360}.$ [26] $x=1$ and $-4.$

[27] $x=5$ and $1.$ [28] $x=8$ and $-2\frac{333}{634}.$

[29] $x=1\pm\sqrt{-2}.$ [30] $x=5$ and $-4\frac{2}{7}.$

[31] $x=1$ and $-\frac{17}{32}.$ [32] $x=\frac{a-2b}{3ab}$ and $\frac{4a-5b}{6ab}.$

[33] $x=\sqrt{a}\pm\sqrt{b}.$ [34] $x=\pm\frac{5}{13}$ or $\pm\sqrt{-1}.$

[35] $x=1$ and $2.$ [36] $x=16$ or $25.$

[37] $x={\cdot}85355$ and ${\cdot}14645.$ [38] $x=(b-a)^2$ and $(+2a)^2.$

[39] $x=(2a-b)^2$ and $(2b-a)^2.$ [40] $x=(+7)^2$ and $(+14)^2.$

EXERCISE LVI.

[1] $x=\pm 1{\cdot}2,\ y=\pm 3{\cdot}4.$ [2] $x=\pm 1,\ y=\pm 2.$

[3] $x=\pm\sqrt{ab},\ y=\pm\sqrt{\frac{a}{b}}.$ [4] $x=\pm 45\dot{6},\ y=\pm 123.$

[5] $x=\pm 8,\ y=\pm 6.$ [6] $x=\pm 15,\ z=\pm 5.$

[7] $x=\pm\frac{a}{\sqrt{(a+b)}}$, $y=\pm\frac{b}{\sqrt{(a+b)}}$. [8] $x=\pm\frac{1}{a-b}$, $y=\pm(a-b)$.

[9] $x=6$, $y=12$, $x'=-14$, $y'=12$, $x''=6$, $y''=2$, $x'''=-14$, $y'''=2$.

[10] $x=\pm 3$, $y=\pm 5$.

[11] $x=3$, $y=5$, $x'=-3$, $y'=5$, $x''=3$, $y''=3$, $x'''=-3$, $y'''=3$.

[12] $x=1$, $y=2$, $x'=3$, $y'=4$.

[13] $x=4$, $y=5$, $x'=4$, $y'=-13$, $x''=-4$, $y''=-5$, $x'''=-4$, $y'''=13$.

[14] $x=\pm 18$, $y=\pm 2$. [15] $\left.\begin{matrix}x\\y'\end{matrix}\right\}=\frac{s+\sqrt{s^2-4p}}{2}$, $\left.\begin{matrix}y\\x'\end{matrix}\right\}=\frac{s-\sqrt{s^2-4p}}{2}$.

[16] $\substack{x\\x'}=d\pm\sqrt{d^2+4p}$, $\substack{y\\y'}=-d\pm\sqrt{d^2+4p}$.

[17] $x=6$, $x'=3$, $y=30$, $y'=6$. [18] $x=1$, $y=4$, $x'=3$, $y'=2$.

[19] $x=\pm 2$, $y=\pm 1$, $x'=\pm\sqrt{4 \cdot 4}$, $y'=\pm\sqrt{1 \cdot 8}$.

[20] $x=15$, $y=4$, $x'=-4$, $y'=-15$.

[21] $x=17$, $y=23$, $x'=62$, $y'=-22$. [22] $x=10$, $y=4$, $x'=12$, $y'=2$.

[23] $x=15129$, $y=119025$, $x'=119025$, $y'=15129$.

[24] $x=7$, $y=24$, $x'=24$, $y'=7$. [25] $x=\frac{1}{4}$, $y=\frac{1}{5}$, $x'=-\frac{1}{4}$, $y'=-\frac{1}{5}$.

[26] $x=4$, $y=9$, $x'=9$, $y'=4$.

[27] $x=0 \cdot 125$, $y=0 \cdot 375$, $x'=0 \cdot 125$, $y'=0 \cdot 625$.

[28] $x=1$, $y=2$, $x'=1\frac{7}{17}$, $y'=-\frac{1}{17}$. [29] $x=\frac{1}{a\pm b}$, $y=\frac{1}{a\mp b}$.

[30] $x=\pm\frac{m+n}{\sqrt{(m+2n)}}$, $y=\pm\frac{n}{\sqrt{(m+2n)}}$.

[31] $x=11$, $y=3$, $x'=-7\frac{1}{2}$, $y'=-3\frac{1}{6}$.

[32] $x=\frac{a}{2}\pm\sqrt{\frac{4b-a^3}{12a}}$, $y=\frac{a}{2}\mp\sqrt{\frac{4b-a^3}{12a}}$.

[33] $x=125$, $y=343$, $x'=343$, $y'=125$.

[34] $x=\pm\sqrt{\frac{4b-a^3}{12a}}+\frac{a}{2}$, $y=\pm\sqrt{\frac{4b-a^3}{12a}}-\frac{a}{2}$.

[35] $x=\pm 125$, $y=\pm 27$. [36] $x=(+5)^2=25$, $y=(+7)^2=49$.

[37] $x=5$, $y=3$, $x'=-3$, $y'=-5$. [38] $x=2$, $y=1$, $x'=1$, $y'=2$.

[39] $x=0$, $y=0$, $x'=\frac{b^2a}{b^2-a^2}$, $y'=\frac{ba^2}{b^2-a^2}$.

[40] $x=\frac{a+1}{1+ab}$, $y=\frac{a(b+1)}{1+ab}$, $x'=\frac{b+1}{1+ab}$, $y'=\frac{b(a+1)}{1+ab}$.

PROBLEMS PRODUCING QUADRATICS.

EXERCISE LVII.

[1] ± 385. [2] $\pm 11\frac{1}{2}$. [3] ± 13.
[4] ± 77 and ± 91. [5] $\pm\sqrt{10}=3 \cdot 162278$. [6] 78, 52, 39.
[7] 481, 259. [8] $\sqrt{\frac{pa}{q}}$ and $\sqrt{\frac{aq}{p}}$, or $-\sqrt{\frac{pa}{q}}$ and $-\sqrt{\frac{aq}{p}}$.
[9] 11 or -24. [10] 8 and 12. [11] 12 feet.

[12] 672 feet long. [13] 960 and 280. [14] 24.
[15] $\frac{2}{5}$. [16] $\frac{4}{25}$. [17] $5\frac{1}{2}$ inches.
[18] The one in 182″, the other in 70″.
[19] 385 feet one, the other 324 feet.
[20] 21 feet long, 13 feet broad, 9 feet high. [21] 343 and 64.
[22] $\frac{5}{7}$ the other value. [23] 16 yards long, 2 yards broad.
[24] 1200 men, each receives $3\frac{3}{4}$ lbs. [25] 28 men, 45 lbs.
[26] 35 and 28. [27] In 4 hours.
[28] 24 lbs. coffee, 330 lbs. sugar. [29] $\frac{1}{16}n^2$.
[30] A 5, B 6, C 7, D 8.

INDETERMINATE EQUATIONS.

EXERCISE LVIII.

[1] In six ways:

guinea pieces	92	75	58	41	24	7
17*s.* pieces	4	25	46	67	88	109

[2] 7 oxen and 60 sheep. [3] 6 men and 9 women. [4] 7691.
[5] 60 values of x and y, according to value of z.

z	y	x
1	4, 9, 14, 19, 24, 29	37, 30, 23, 16, 9, 2
2	1, 6, 11, 16, 21, 26	39, 32, 25, 18, 11, 4
&c.	&c. &c. &c.	&c. &c. &c.
18	3	1

[6] $x=1, 2, 3, 4, 10, 11, 12, 13, 14, 15$.
$y=8, 6, 4, 2, 11, 9, 7, 5, 3, 1$.
$z=11, 12, 13, 14, 1, 2, 3, 4, 5, 6$.
[7] 127 feet long. [8] I give him 11 crowns, he gives me 10 pieces.

ARITHMETICAL PROGRESSION.

EXERCISE LIX.

[1] *l.* 50, *s.* 442. [2] *l.* $10\frac{3}{4}$, *s.* 142. [3] *l.* $35\frac{1}{2}$, *s.* 1900.
[4] *l.* $3\frac{7}{8}$, *s.* $60\frac{1}{8}$. [5] *l.* $20\frac{5}{7}$, *s.* $139\frac{2}{7}$. [6] *l.* 14, *s.* 28.
[7] *l.* $-15\frac{3}{4}$, *s.* $-146\frac{1}{4}$. [8] *l.* $-1\frac{7}{8}$, *s.* $-13\frac{3}{4}$. [9] *l.* -10, *s.* -90.
[10] *l.* $-2\frac{3}{4}$, *s.* $-281\frac{1}{4}$.

EXERCISE LX.

[1] 2. [2] 32. [3] 300. [4] -31, -240.
[5] -129, -1250. [6] -2, -6, -10, -14, -18, -22.
[7] $\frac{1}{3}$, $\frac{1}{6}$, 0, $-\frac{1}{6}$, $-\frac{1}{3}$, $-\frac{1}{2}$, $-\frac{2}{3}$. [8] 24, 31, 38, 45. [9] 1, $3\frac{1}{2}$, 6, $8\frac{1}{2}$.
[10] 16, 12, 8, 4. [11] 16, 20. [12] 3, 5, 7, 9, 11.
[13] 275*l.* 12*s.*, 10*l.* 8*s.* last payment. [14] $6\frac{1}{44}$ miles.

GEOMETRICAL PROGRESSION.

EXERCISE LXI.

[1] *l.* 64, *s.* 127. [2] *l.* 78732, *s.* 118096.
[3] *l.* 327680, *s.* 436905. [4] *l.* $258\frac{2073}{4096}$, *s.* $591\frac{741}{4096}$.
[5] *l.* $106\frac{403}{512}$, *s.* $307\frac{441}{512}$. [6] *l.* $1\frac{317}{512}$, *s.* $19\frac{373}{512}$.
[7] $15\frac{2047}{2048}$. [8] $2\frac{166207}{354294}$. [9] 33741·59 . . . [10] $8\frac{45237}{78125}$.
[11] 235125·85 . . . [12] 30570·01310 . . .

EXERCISE LXII.

[1] $\frac{2}{3}$. [2] 16. [3] $\frac{1}{2}$. [4] 2. [5] 70. [6] 27. [7] $2\frac{11}{17}$.
[8] $1\frac{1}{7}$. [9] $\frac{2}{9}$. [10] $\frac{4}{9}$. [11] $\frac{1}{3}$. [12] $\frac{2}{11}$. [13] $\frac{32}{99}$. [14] $\frac{23}{99}$.

EXERCISE LXIII.

[1] 13 terms. [2] 3, 12, 48, 192. [3] Let S = half the sum, and D half the difference of two means, then $S=\frac{-b\pm\sqrt{b^2+2a^2(a^2-b)}}{4a}$, and $D=S\sqrt{\frac{a-4S}{a+4S}}$. [4] 3, 6, 12, 24, 48. [5] $r=\pm\frac{\sqrt{4b-3a}-\sqrt{a}}{2\sqrt{a}}$.

[6] 4, 8, 16, 32; $\frac{1}{3}$, 1, 3, 9. [7] ·9439, ·8909, ·8409, ·7937, ·7492, ·7071, ·6659, ·6300, ·5946, ·5612, ·5297. [8] $r=1·0935$, $u=27·351$.

[9] $\frac{2}{5}$, $\frac{4}{25}$, $\frac{8}{125}$, &c. [10] Let a, ar, ar^2, ar^3 be the four quantities; then $a+ar^3>ar+ar^2$, if $1+r^3$ be $>r+r^2$ or $r(1+r)$, or if $1-r+r^2>r$, or if $1-2r+r^2=(1-r)^2$ be >0. But every square is positive, and ∴ >0, ∴ $a+ar^3>ar+ar^2$.

HARMONICAL PROGRESSION.

EXERCISE LXIV.

[1] (i) $1\frac{1}{7}$, $1\frac{1}{3}$, $1\frac{3}{5}$, 2, $2\frac{2}{3}$, 4, 8, ∞, −8; (ii) $\frac{6}{7}$, 1, $1\frac{1}{5}$, $1\frac{1}{2}$, 2, 3, 6, ∞, −6; (iii) $\frac{3}{32}$, $\frac{3}{29}$, $\frac{3}{26}$, $\frac{3}{23}$, $\frac{3}{20}$, $\frac{3}{17}$, $\frac{3}{14}$, $\frac{3}{11}$, $\frac{3}{8}$. [2] $\frac{1}{3}$, $\frac{1}{5}$, $\frac{1}{7}$, and 2, $2\frac{2}{3}$, 4. [3] 6 and 2.

EXERCISE LXV.

[1] $4\frac{1}{4}$, 2, $\frac{16}{17}$. [2] 10, 6, $3\frac{3}{5}$. [3] $3\frac{1}{8}$, 3, $1\frac{11}{35}$. [4] 20.
[5] −16. [6] 42. [7] $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$. [8] 104 and 234.

RATIOS.

EXERCISE LXVI.

[1] $\frac{35}{42}:\frac{24}{42}$, $\frac{88}{216}:\frac{153}{216}$, $\frac{63}{144}:\frac{88}{144}$. [2] $a+b:b$, $4a+5:5a+4$, $5x-3:x+1$, $1:x+1$. [3] 3 : 4, 22 : 43, 9 : 11.
[4] 3 : 1, 7 : 5, 315 : 44. [5] 4 : 1.
[6] $a^3-8ax^2+3x^3:a^2-2ax-3x^2$. [7] 8 : 189. [9] 33 : 65.
[10] $614\frac{1}{8}:512$. [11] 490 millions of miles. [12] About 30 years.

PROPORTION.

EXERCISE LXVII.

[1] 12, $24\frac{1}{2}$, $20\frac{1}{4}$, $\frac{27}{32}$. [2] 6, $10\frac{2}{7}$, 1. [3] 10, 9, $1\frac{1}{2}$.
[7] $x=7$. [8] $x=1$, $y=3$.
[9] $x=\frac{md(ph-fq)-eqnb}{(na-mc)(ph-fg)-pgnb}$, $y=\frac{eq(na-mc)-mdpq}{(na-mc)(ph-fg)-pgnb}$.
[10] $x=(a^3+b^3)$, $y=(a^3-b^3)$. [11] $x=a^4-1$, $y=a^3-1$, $z=a^2-1$, $u=a-1$. [12] 38 : 76 : 152. [13] 8 days.
[14] $\frac{pn}{q-p}$ seconds after the second sets out. [15] 20 years. [16] 300.

VARIATION.

EXERCISE LXVIII.

[1] $y=\frac{3}{2}x$. [2] $y=\frac{4}{x}$. [3] $x=\frac{1}{24}yz^2$.
[5] $x=2y+y^2$. [6] $x=5-\frac{5}{4}y+\frac{3}{4}y^2$.

PERMUTATIONS, COMBINATIONS.

EXERCISE LXIX.

[1] 12. [2] 2520, 840. [3] 90, 1801800.
[4] 73513440, 30270240. [5] 1, m. [6] $\frac{m(m-1)}{1\,.\,2}$, $\frac{m(m-1)(m-2)}{1\,.\,2\,.\,3}$.
[7] $\frac{m(m-1)(m-2)(m-3)}{1\,.\,2\,.\,3\,.\,4}$, $\frac{m(m-1)(m-2)(m-3)(m-4)}{1\,.\,2\,.\,3\,.\,4\,.\,5}$.
[8] 1814400. [9] 720. [10] 6. [11] 11.
[12] 360, 5040, 840, 100800. [13] 8. [14] 2.

EXERCISE LXX.

[1] 720. [2] 153. [3] 6561. [4] 56. [5] 210.
[6] 4845. [7] 140. [8] 1679616. [9] 4096.

BINOMIAL THEOREM FOR POSITIVE WHOLE EXPONENTS.

EXERCISE LXXI.

[1] $a^6 \pm 6a^5b+15a^4b^2 \pm 20a^3b^3+15a^2b^4 \pm 6ab^5+b^6$.

[2] $a^9 \pm 9a^8b+36a^7b^2 \pm 84a^6b^3+126a^5b^4 \pm 126a^4b^5+84a^3b^6 \pm 36a^2b^7+9ab^8 \pm b^9$.

[3] $1 \pm 11x+55x^2 \pm 165x^3+330x^4 \pm 462x^5+462x^6 \pm 330x^7+165x^8 \pm 55x^9+11x^{10} \pm x^{11}$.

[4] $625-2000x+2400x^2-1280x^3+256x^4$.

[5] $729-2946x^2+4860x^4-4320x^6+2160x^8-576x^{10}+64x^{12}$.

[6] $\frac{1}{128}x^7 + \frac{7}{32}x^6y + \frac{21}{8}x^5y^2 + \frac{35}{2}x^4y^3 + 70x^3y^4 + 168x^2y^5 + 224xy^6 + 128y^7$.

[7] $a^{27} + 27a^{25}b + 324a^{23}b^2 + 2268a^{21}b^3 + 10206a^{19}b^4 + 30618a^{17}b^5 + 61236a^{15}b^6 + 78732a^{13}b^7 + 59049a^{11}b^8 + 19683a^9b^9$.

[8] $243a^5c^5 - 810a^4c^4bd + 1080a^3c^3b^2d^2 - 720a^2c^2b^3d^3 + 240acb^4d^4 - 32b^5d^5$.

[9] $625a^8c^8d^4 - 2000a^7bc^6d^5 + 2400a^6b^2c^4d^6 - 1280a^5b^3c^2d^7 + 256a^4b^4d^8$.

[10] $64a^6c^6b^{-12} + 48a^5c^7db^{-9} + 15a^4c^8d^2b^{-6} + \frac{5}{2}a^3c^2d^3b^{-3} + \frac{15}{64}a^2c^{10}d^4 + \frac{3}{256}ab^3c^{11}d^5 + \frac{1}{4096}b^6c^{12}d^6$.

[11] $a^2+6ab+b^2 \pm (4a+4b)\sqrt{ab}$.

[12] $(a^3+21a^2b+35ab^2+7b^3)\sqrt{a} \pm (7a^3+35a^2b+21ab^2+b^3)\sqrt{b}$.

[13] $2\left\{a^n + \frac{n(n-1)}{1\cdot 2}a^{n-2}b^2 + \frac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4}a^{n-4}b^4 + \frac{n\,..\,n-5}{1\,..\,6}a^{n-6}b^6 + \,..\right\}$

[14] $2\left\{\frac{n}{1}a^{n-1}b + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}a^{n-3}b^3 + \frac{n\,..\,n-4}{1\,..\,5}a^{n-5}b^5 + \frac{n\,..\,n-6}{1\,..\,7}a^{n-7}b^7 + \,..\right\}$

EXERCISE LXXII.

[1] $a^n - \frac{n(n-1)}{1\cdot 2}a^{n-2}b^2 + \frac{n\,..\,n-3}{1\,..\,4}a^{n-4}b^4 - \ldots$

$\pm\left\{\frac{n}{1}a^{n-1}b - \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}a^{n-3}b^3 + \frac{n\,..\,n-4}{1\,..\,5}a^{n-5}b^5 - \ldots\right\}\sqrt{-1}$.

[2] $2\left\{a^n - \frac{n(n-1)}{1\cdot 2}a^{n-2}b^2 + \frac{n\,..\,n-3}{1\,..\,4}a^{n-4}b^4 - \ldots\right\}$

[3] $2\left\{na^{n-1}b - \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}a^{n-3}b^3 + \frac{n\,..\,n-4}{1\,..\,5}a^{n-5}b^5 - \ldots\right\}$

[4] $105a^{13}b^2$. [5] $1820a^{12}b^4$. [6] $-142506a^{25}b^5$. [7] $161700a^{97}b^5$.

[8] $495a^{16}b^8$. [9] $192192a^6b^6c^8d^8$. [10] $12870a^8b^8$.

[11] $-48620a^9b^9$. [12] $24310a^9b^8 - 24310a^8b^9$.

[13] $-92378a^{10}b^9 + 92378a^9b^{10}$.

GENERAL BINOMIAL THEOREM.

EXERCISE LXXIII.

[1] $a^{\frac{7}{4}} + \frac{7}{4}a^{\frac{3}{4}}b + \frac{21}{32}a^{-\frac{1}{4}}b^2 - \frac{7}{128}a^{-\frac{5}{4}}b^3 + \frac{35}{2048}a^{-\frac{9}{4}}b^4 - \ldots$

[2] $x^{\frac{2}{3}} - \frac{2}{9}x^{-\frac{7}{3}}z + \frac{7}{81}x^{-\frac{16}{3}}z^2 - \frac{112}{2187}x^{-\frac{25}{3}}z^3 - \ldots$

[3] $1 \pm \frac{1}{3}z - \frac{1}{9}z^2 \pm \frac{5}{81}z^3 - \frac{10}{243}z^4 \pm \ldots$

[4] $1 \pm \frac{1}{4}z - \frac{3}{32}z^2 \pm \frac{7}{128}z^3 - \frac{77}{2048}z^4 \pm \ldots$

[5] $1 + 4x + 10x^2 + 20x^3 + 35x^4 + \ldots$

[6] $1 - 2x + 3x^2 - 4x^3 + 5x^4 - \ldots$

[7] $1 + 3x + 6x^2 + 10x^3 + 15x^4 + \ldots$

[8] $1 \pm \frac{1}{2}z - \frac{1}{8}z^2 \pm \frac{1}{16}z^3 - \frac{5}{128}z^4 \pm \ldots$

[9] $\sqrt[4]{x}\left\{1 - \frac{1}{4}\cdot\frac{a}{x} + \frac{5}{32}\cdot\frac{a^2}{x^2} - \frac{15}{128}\cdot\frac{a^3}{x^3} + \frac{195}{2048}\cdot\frac{a^4}{x^4} - \ldots\right\}$

[10] $\frac{1}{3a}\left\{1-\frac{10b}{9a}+\frac{125b^2}{81a^2}+\frac{5000b^3}{2187a^3}+\frac{68750b^4}{19683a^4}+\ldots\right\}\sqrt[3]{3a}$.

[11] $m^{-\frac{3}{4}}+\frac{3}{4}m^{-\frac{7}{4}}n+\frac{21}{32}m^{-\frac{11}{4}}n^2+\frac{77}{128}m^{-\frac{15}{4}}n^3+\ldots$

[12] $1-\frac{1}{3}x^{-3}y^3+\frac{2}{9}x^{-6}y^6-\frac{14}{81}x^{-9}y^9+\frac{35}{243}x^{-12}y^{12}-\ldots$

[13] $a^{\frac{10}{3}}-\frac{2}{3}a^{-\frac{4}{3}}x^5+\frac{2}{8}a^{-\frac{10}{3}}x^{25}$. [14] $14a^{-\frac{1}{3}}x^{\frac{1}{3}}+\frac{56}{3}a^{-\frac{4}{3}}bx^{\frac{1}{3}}-\frac{280}{9}a^{-\frac{7}{3}}b^2x^{\frac{1}{3}}$.

[15] $x^{-12m}-6x^{-14m}y^{-2n}+21x^{-16m}y^{-4n}-56x^{-18m}y^{-6n}$.

[16] $x^{-30}+10x^{-35}y^{-2}+55x^{-40}y^{-4}+220x^{-45}y^{-6}$.

[17] $2-\frac{1}{3.2^2}-\frac{1}{3.6.2^4}-\frac{5}{3.6.9.2^7}-\frac{5.8}{3.6.9.12.2^{10}}-\ldots$

[18] $2\left\{1+\frac{1}{2^3}-\frac{1}{2^7}+\frac{1}{2^{10}}-\frac{5}{2^{15}}+\ldots\right\}$.

[19] $7\left\{1+\frac{1}{2}\cdot\frac{1}{49}-\frac{1}{8}\cdot\frac{1}{49^2}+\frac{1}{16}\cdot\frac{1}{49^3}-\ldots\right\}$.

[20] $N\left\{1+\frac{1}{p}\cdot\frac{x}{N^p}+\frac{\frac{1}{p}\left(\frac{1}{p}-1\right)}{1\,.\,2}\cdot\frac{x^2}{N^{2p}}+\ldots\right\}$.

NOTATION.

Exercise LXXIV.

[1] 232320, 44000, 21520, 11514, 5670. [2] 53456811.

[3] 100100010, 10202, 1202. [4] 237, 554. [5] 28024, 530.

[6] 2010100123. [7] 3200014, 540677122.

[8] 326928534. [9] 5120.

[10] 102120 and 222. sum, 103002. dif. 101232. prod. 30003300. quot. 130. [11] 3211, 75944, 7761. [12] 674, 2020, and 3234.

Miscellaneous Exercises.

[1] $(b^2-c^2)a^3+(b^4-c^4)a^2+3(b+c)a$. [2] $7a-13d+24c-5f$.

[3] $\frac{b^2}{a+b}$. [4] $\frac{3c(a+b)^2(c^2+x^2)^{n-5}}{a^3b^3}$. [5] $x=2$.

[6] $\frac{3}{2(a+b)^2}$ and $(a+x)^6(a+y)^4$. [7] $\frac{a}{1+m+n}$, $\frac{ma}{1+m+n}$, $\frac{na}{1+m+n}$.

[8] $\frac{\sqrt[4]{a^3}-\sqrt[4]{a^2b}+\sqrt[4]{ab^2}-\sqrt[4]{b^3}}{a-b}$ and $\frac{a+b+2\sqrt{ab}+2\sqrt[4]{a^3b}+2\sqrt[4]{ab^3}}{a-b}$.

[9] Let d be the difference of the middle terms,
then $d=\sqrt{\{-a^2-2b^2\pm2\sqrt{(c+2a^2b^2)}\}}$, and the proportion is
$\frac{1}{2}\{b-\sqrt{(b^2-a^2+d^2)}\}:\frac{1}{2}(a-d)=\frac{1}{2}(a+d):\frac{1}{2}\{b+\sqrt{(b^2-a^2+d^2)}\}$.

[10] $(2a^pb^n+3a^{2-n}b-c^2)\sqrt[m]{a^3b^5}$.

[11] 1·32287 . . , 3·20936 . . , ·11832. . [12] $\frac{ma+l}{m+1}$, $\frac{(m-1)a+2l}{m+1}$, &c. $\frac{2a+(m-1)l}{m+1}$, $\frac{a+ml}{m+1}$. [13] $\sqrt{(1-x)}$ and $\frac{1}{\sqrt{(a^2+x^2)}}$. [14] $\frac{2}{3}$.

[15] $x^3+(a+b+c)x^2-(ab-ac-bc)x-abc$ and $x^4+(a+b+c+d)x^3+(ab+ac+ad+bc+bd+cd)x^2+(abc+abd+acd+bcd)x+abcd$.

[16] $\frac{2}{5}$. [17] $-\frac{112}{2187}x^{\frac{4}{3}}y^3$, $a^{-\frac{3}{4}}+\frac{3}{4}a^{-\frac{7}{4}}b+\frac{21}{32}a^{-\frac{11}{4}}b^2+\frac{77}{128}a^{-\frac{15}{4}}b^3+\ldots$
[18] $(+3\frac{1}{3})^2=11\frac{1}{9}$, $(-8)^2=64$, $(+10)^2=100$. [19] $\frac{a^2}{a+b}$.
[20] $\frac{86}{99}$, $\frac{125}{333}$. [21] $\frac{x^2-5ax+4a^2}{x-3a}$, $\frac{a^2+x^2}{a-x}$. [22] 7, 16, 25, 42.
[23] 24, −115. [24] $1+1+\frac{5}{9}+\frac{7}{27}+\frac{9}{81}+\ldots$
[25] $4\frac{1}{2}a^3-3a^2x+1\frac{1}{2}ax^2-7x^3$. [26] $2x^3-3x^2-x+5$.
[27] $\pm 5x^2-1$, $x^2-(a-b)x+b^2$. [28] 45 shillings.
[29] $\frac{a\surd b}{b}$, $\frac{\surd mn}{n}$, $\frac{m(n-\surd p)}{n^2-p}$, $\frac{a(\surd m\mp\surd n)}{m-n}$. [31] $x^{m-n}y^{m+n}$.
[32] $-71145x^9$. [33] 1663200.
[34] $a+\frac{x^2}{2a}-\frac{x^4}{8a^3}+\frac{x^6}{16a^5}-\frac{5x^8}{128a^7}$, &c. and $1-\frac{x^3}{3}-\frac{x^6}{9}-\frac{5x^9}{81}$, &c.
[35] 1920. [36] $\frac{1}{1-x^3}$, $\frac{a+cb^r+db^{2r}}{b^n}$. [38] $7414\frac{7}{8}$ the one, 4913 the other.
[39] (i) $x=3$, $y=5$, or $x=5$, $y=3$, or $x=\sqrt[3]{(76\pm 4\sqrt{617})}$, $y=\sqrt[3]{(76\mp 4\sqrt{617})}$; (ii) $x=\pm\sqrt{\{a^2-b+\sqrt{(b^2+c^2)}\}}$ and $\pm\sqrt{\{a^2-b-\sqrt{(b^2+c^2)}\}}$. [40] 39 and 539, 507 and 7. [41] $\frac{1+x+x^2}{1-x-x^4+x^5}$.
[42] 36. [43] $\sqrt{ab}-\sqrt{(a^2-b^2)}$ and $2-\sqrt{-3}$.
[44] $\sqrt{(x-1)}-1$ and $\frac{c+\surd(a^2-c^2)}{2}$. [45] 3·0455 and $\frac{3b^m u^3 x^4}{2h^3(a^2+x^2)^{2m}}$.
[47] $\sqrt[mnp]{a}$, $\sqrt[3]{3a}$, $\sqrt[4]{2b}$, $\sqrt[24]{64a^6b^2c^2d}$. [49] $\frac{p\surd b}{\surd(p^2-q^2)}$, $\frac{q\surd b}{\surd(p^2-q^2)}$.
[50] 8. [51] $m^{\frac{1}{2}}\times m^{\frac{1}{2}}$, $n^{\frac{1}{2}}\times m^{\frac{1}{2}}$, $n^{\frac{1}{3}}\times m^{\frac{1}{2}}$, $n^{\frac{1}{2}}\times m^{\frac{1}{3}}$, $n^{\frac{1}{2}}\times m^{\frac{1}{3}}$, $n^{\frac{1}{2}}\times m^{\frac{1}{2}}$.
[52] $8\sqrt[3]{2}$, $\frac{28}{15}\sqrt{15}$. [53] $3\frac{11}{45}\sqrt[3]{6}$, $\frac{4b-a}{3ab^2}\sqrt{3b}$.
[54] $2pq(108p^{\frac{1}{2}}q^{\frac{1}{3}}r^{\frac{1}{12}})$, $\frac{a^{\frac{3m+2}{24}}}{b^{\frac{1}{6}}c^{\frac{1}{12}}}$. [55] $2\sqrt{3}$, $4\sqrt[6]{32}-6\sqrt{6}-6\sqrt[3]{6}+9\sqrt[6]{243}$.
[56] $362+209\sqrt{3}$ and $1\frac{3}{8}\sqrt{7}-1\frac{31}{56}\sqrt{3}$.
[57] $\sqrt{2(a\pm\sqrt{a^2-b})}$, $2(a\pm\sqrt{a^2-b})$.
[58] $\sqrt[4]{18}+\sqrt[4]{2}$, $\sqrt[4]{43\cdot 75}-\sqrt[4]{1\cdot 75}$. [59] $\frac{3x^3+x^2+1}{5x^2+x+2}$. [60] $\frac{2}{3}$.
[61] 1·105. [62] 5 and 3, 31 and 20, 57 and 37, &c.
[63] $1-\frac{12}{5}+\frac{12.13}{1.2}\cdot\frac{1}{5^2}-\frac{12.13.14}{1.2.3}\cdot\frac{1}{5^3}+\ldots=1-\frac{300}{125}+\frac{390}{125}-\frac{364}{125}+$ &c., the third is therefore the greatest term. [64] $\frac{20hx-22x^2}{(h+x)(h-2x)^2}$.
[65] $\frac{3a^{2n-1}bc^r}{2d^2}$, $\frac{a+b}{(c+a-b)(b-a+c)}$. [66] 16. [67] $\frac{1}{5}$. [68] 56.
[69] (i) 1, (ii) $\frac{a^3-b^3}{3a}$. [70] $2\left\{1+\frac{3}{5.32}-\frac{4.3^2}{2.5.(32)^2}+\frac{4.9.3^3}{2.3.5^3(32)^3}, \text{\&c.}\right\}$
[71] $x+1$; $2x-7$. [72] $x^5-x^3y^2-x^2y^3-y^5$; $3p^3q+24p^2q+45pq^3$.
[73] $6\frac{2}{3}$. [74] $4924\frac{7}{10}l$. [75] 24 : 30 : 21 : 70 : 18 : 28.
[76] 204 feet long, 85 feet broad. [77] $x^2-2(a+b)x+(a+b)^2+c^2=0$.
[78] Lessen both terms by 3. [79] $2+\sqrt{-3}-4\sqrt{-5}$. [80] $789\frac{10}{11}$.

[81] Both are positive when a and b of the equation x^2-ax+b are positive. In this case $\frac{1}{2}a > \sqrt{\frac{1}{4}a^2-b}$. They are both negative when a is $-$ and b $+$, in which case the equation is of the form $x^2+mx+n=0$. The greater root will be $-$ and the less $+$ when a and b are both $-$, the greater will be $+$ and the less $-$ when a is $+$ and b $-$. In the former case the equation is of the form $x^2+mx-n=0$, in the latter, of the form $x^2-mx-n=0$. [82] 72. [83] 2 hours $43\frac{7}{11}$ min. [84] $a^{\frac{1}{3}}$.

[85] 18 and 9. [86] 560 and 420. [87] $x=\frac{2a}{\sqrt{3}}$.

[88] $x=a\left(\frac{b+a}{b-a}\right)^2$. [89] $\frac{5}{6}$. [90] $x=\left(\frac{m+n}{2}\right)^{\frac{1}{2}}$, $y=\left(\frac{m-n}{2}\right)^{\frac{1}{2}}$.

[91] 6 days. [92] 3540. [93] $x^3+\frac{1}{x^3}+x+\frac{1}{x}$. [94] $\sqrt{x}+\sqrt{y}$.

[95] 160 gallons. [96] 10 days. [97] 50. [98] 12 feet.

[99] $x=a\left(\frac{m}{a^2+b^2}\right)^{\frac{1}{2}}$, $y=b\left(\frac{m}{a^2-b^2}\right)^{\frac{1}{2}}$. [100] 40 and 50.

[101] $2x+\frac{cx-bx+ab}{bc}$. [102] $3\sqrt[3]{5}$ and $\frac{1}{12}\sqrt[3]{18}$. [103] 125250.

[104] $x=\frac{a}{3}$. [105] 40 worked, 10 idle. [106] $x=\frac{a^2+b}{2a}$, $y=\frac{a^2-b}{2a}$.

[107] (i) $x^4-8x^3+24x^2-32x+16$, (ii) $a^3-3a^2b+3a^2c+3ab^2-6abc+3ac^2-b^3+3b^2c-3bc^2+c^3$, (iii) $1+6x+15x^2+20x^3+15x^4+6x^5+x^6$.

[108] 60 shillings. [109] $x+1$, x^2-b^2, $x+b$, x^2-y^2.

[110] A$=$40, B$=$5, C$=$120. [111] x^2+1, $1+y$. [112] 161 and 30.

[113] $6(a^2-b^2)$. [114] $\frac{x}{x^2+xy+x-y}$. [115] $ab^2(a^2-b^2)$.

[116] $x=\sqrt{\frac{n^4-m^4}{2m^2}}$. [117] $x=\frac{p}{3}$. [118] 72. [119] 100.

[120] $13\frac{1}{3}$. [121] 500. [122] 51 gallons brandy, 14 British spirits.

[123] A in $14\frac{34}{49}$, B in $17\frac{23}{41}$, C in $23\frac{7}{81}$. [124] 18, 30, 4, 144.

[125] $x=(\frac{1}{2}\pm\sqrt{a+\frac{1}{4}})^2$. [126] $x=\sqrt{(\frac{1}{3}\sqrt{76}+\frac{1}{3})}^{\frac{1}{n}}$. [127] 20.

[128] 16. [129] 5, 9, 13, 17. [130] 3, 6, 12. [131] 50.

[132] 12. [133] 3, 6, 9, 12. [134] 1, 6, 11, 16.

[135] $\sqrt{1-x}$, $\frac{1}{\sqrt{(a^2+x^2)}}$. [136] $p-2q+3r$. [137] 1·93098, 2·0928.

[138] 11, 9. [139] $7\frac{1}{2}$, 9, $10\frac{1}{2}$, 81, 243, 729. [140] $x=2n$.

[141] $\frac{1}{(1-x)^3}$, $\frac{1}{2}$. [142] $\frac{5^{\frac{3}{2}}-5^{\frac{1}{2}}3^{\frac{1}{2}}+3^{\frac{1}{2}}5^{\frac{1}{2}}-3^{\frac{3}{2}}}{2}\sqrt{ab}$. [143] $3\frac{1}{2}$ years ago.

[144] $(3a-b-1)x^3$. [145] $\frac{3x+2y}{3y-2x}$. [146] $\frac{x^4}{x^{\frac{1}{2}}+b}$.

[147] The n^{th} term is $\frac{5}{6}-\frac{1}{3}n$, and the sum of n terms is $\frac{2}{3}n-\frac{1}{6}n^2$.

[148] $\frac{2b}{1-a}$, $\frac{2b}{1+a}$. [149] $1\frac{1}{2}$, $\frac{1}{2}$. [150] $\frac{4a}{a^2-b^2}$, $\frac{4b}{a^2-b^2}$.

[151] $x=25$, $y=20$. [152] 73, 9, 328.

[153] The exponent is 1·0935, the sum 881·62, the 20th term 27·351.

[154] $x=22$. [155] ·08882 . ., 2·50222 . . .

[156] $\frac{a^2-b+2a\sqrt{-b}}{a^2+b}$ and $\sqrt[n]{\{a+\sqrt{(a^2-b^n)}\}}$. [157] 6. [158] a^n.

[159] 308*l.* 12*s.* 10*d.* [160] $x=1$, $y=4$. [161] x^{5m-6}, $\sqrt[m]{\frac{ab}{c}}$.
[162] 720 lbs., 512 lbs. [163] 6 terms. [164] 142 years.
[165] 16 years. [166] 7·1773. [167] $x^2-2(a+b)x+(a+b)^2+c^2=0$.
[168] $789\frac{10}{11}$. [169] $x=12$, $y=30$, $z=168$, $u=50$.
[170] $1+\frac{1}{2}-\frac{1}{2.4}+\frac{3}{2.4.6}-\frac{3.5}{2.4.6.8}+\frac{3.5.7}{2.4.6.8.10}$, &c.
[171] $x=1$, $y=3$, $z=26$. [172] 1770, 34220, 487635, 5461512.
[173] $\frac{1}{2}a^4-2a^3b$, $\frac{a^2c}{b^2}-\frac{2a^3c}{b}-4a^4c$. [174] $\sqrt[24]{a^{41}}$, $a^n\sqrt[x]{a^m}$, $a^4\sqrt[3]{a^2}$.
[175] $\frac{a\sqrt[z]{ab^2c^3d^4}}{bcd}$, a^{7x-8y}. [176] $\frac{x-y-z}{x-4y+4z}$. [177] 13579.
[178] $a^3-a^2b-ab^2+b^3$. [179] 140, 117480, 2555190.
[180] 4 and 6. [181] $\frac{a+\sqrt{(a^2-4b)}}{2}$, $\frac{a-\sqrt{(a^2-4b)}}{2}$.
[182] $\frac{5a^2-b}{4a}$, $2n$. [183] $\frac{5}{18}$, $\frac{n}{m}$. [184] $\left(\frac{n-su}{p-ut}\right)^m+r$, 1000.
[185] $\frac{16a}{9}$. [186] 27 gallons. [187] $\frac{pq}{p+q}$.
[188] Gold $\frac{bpr-aps}{qr-ps}$, silver $\frac{aqr-bpr}{qr-ps}$. [189] 56609, root $=243$.
[190] They travel 56 hours; one goes 84, the other 70 miles.
[191] 80*l.* [192] 18 persons. [195] $\frac{1}{2}$, $1\frac{1}{2}$, $4\frac{1}{2}$. [196] 16 oxen.
[197] 432. [198] 221 : 273 = 187 : 231.
[199] 10 feet broad and 14 feet long. [200] 17.
[201] 74 lbs. tin and 46 lbs. lead. [202] 12 yards and 24 yards.
[203] 82 and 105. [204] 26548 men.
[205] $\frac{d(\sqrt{b}+\sqrt{a})}{\sqrt{b}-\sqrt{a}}$, or 150 miles.
[206] $\left.\begin{matrix}x\\x'\end{matrix}\right\}=p\frac{n\sqrt{(m+p)(m-p)}\pm p\sqrt{(m+n)(m-n)}}{(n+p)(n-p)}$.
$\left.\begin{matrix}y\\y'\end{matrix}\right\}=n\frac{n\sqrt{(m+p)(m-p)}\pm p\sqrt{(m+n)(m-n)}}{(n+p)(n-p)}$.
[207] 8565126197851151797861440000 ways. [208] 0. [209] 2, 3.
[210] $\log. A=\left\{\log. v+\log. (R-1)-\log. \left(1-\frac{1}{R^n}\right)\right\}=1\cdot847336$, and $\therefore$ $A=70\cdot36l.=70l.$ $7s.$ $2d.$

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